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Initial-state heavy quarks in hadronic processes

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Contents

Introduction

Inclusion of heavy quark mass effects is an important factor in modern studies of parton distribution. In fact, working on a wide range of perturbative scales Q^2, precise results require the incorporation of heavy quark mass effects near the threshold $Q^2 \sim m^2$, and the resummation of collinear logarithms at higher scales. These two different kind of accuracy are obtained through two different factorization schemes, called respectively massive and massless.

Depending on the scale of energy of the specific problem we are talking about, it could be convenient to use one between the massive or massless schemes: while the former is accurate near the threshold for heavy quark production, the latter is better at higher energies. Performing computations in the massive scheme, the heavy quarks are treated as massive objects, therefore the full mass dependence is retain. On the other hand, logarithmic terms of Q^2/m^2 explicitly appear in the results, raising from the hard subprocesses where an heavy quark and anti-quark are collineary emitted. In the case of massless quarks, these divergent terms are usually factorized and included in the definition of parton distribution functions (PDF), so that the latter acquire a dependence from the scale of the process, described by the Altarelli-Parisi equations (DGLAP equations). By solving DGLAP equations, we find the exact expression for the PDFs, which takes into account the logarithmic terms resummed to all orders.

Therefore, dealing with problems concerning heavy quarks we are allowed to choose wether to consider them as massive or massless object: in the first case we will retain the full mass dependence, but we will have unresummed logarithms which could become large at higher energy, spoiling our results; in the second case we will obtain results where the huge logarithms are resummed to all orders, keeping a better precision at high energy, but we loose every mass correction terms, which may well be important at lower energies.

Therefore it could be useful to find out a general method giving results which are valid at a generic energy scale, retaining both the mass correction terms coming from the massive scheme and the resummation of the collinear logarithms at higher scale. There are several methods which allow to do this, including heavy quark mass effect, one of which is the so-called FONLL scheme. It was first introduced in the context of hadroproduction, but it has been applied to other QCD processes involving heavy quarks, such as deep-inelastic lepton-hadron scattering, Ref. [1], [2] and hadronic processes, as Higgs production in bottom quark fusion, Ref. [6], [7]. The FONLL method relies on standard QCD factorization, and its implementation only required calculations in well defined factorization schemes, where heavy quarks is either consider as massive or massless partons.

The FONLL method was first introduced with the assumption of perturbative heavy quarks, namely assuming that the latter are generated by radiation from light flavours, and not intrinsically contained in the proton, Ref. [1]. This hypothesis has then been relaxed in Ref. [2], [5], where, in the context of deep inelastic scattering, a formalism including a possible contribution from an intrinsic charm quark in the proton has been developed. This is useful because of several reasons, of both principle and practical nature, and in the specific case of charm quark it could be particularly interesting, because of arguments suggesting that the proton contains a significant charm component. However, also for hadronic processes involving heavy quarks different from the charm, such generalization of the FONLL scheme may well be relevant. In fact in the standard massive scheme, heavy flavours are not considered in the initial state, since they are supposed to be of perturbative origin. Therefore, in the massive scheme, there aren't PDFs referred to them which instead do appear in the massless scheme, just as for the light quarks. The heavy quark PDFs of the massless scheme are defined starting from vanishing boundary condition at some non-perturbative scale m_0, and their general expression at an higher scale Q^2 is worked out by solving DGLAP equations. In this approach we will obtain results generally dependent from the choice of m_0, and this dependence may well be source of bias and errors. Accounting for an intrinsic heavy quark, even if the heavy quarks are indeed of perturbative origin, we would be able to reabsorb the dependence on m_0 in the initial condition, which would be obtained by a fit just as the light flavour PDFs. In addition to this, we wouldn't have the uncertainty due to the fixed order matching between massive and massless scheme, which is another possible source of bias in computation at low orders. Furthermore, in processes where an intrinsic component of heavy quark does exist, this formalism would allow to consider also the initial parametrization for the massive quark content of the initial state.

The aim of the present work is to present and discuss a formalism which generalizes FONLL scheme, including contributions due to intrinsic heavy quarks. We will present first some general features of such method and then we will specify the calculation to the case of Higgs production in bottom quark fusion, finding out the relevant modifications to the results of Ref. [6].

In section 1, the basic formalism regarding the different factorization schemes used for fixed and resummed results is described, and the FONLL method is briefly introduced, as a possible way to match these two schemes. Then, in section 2, we present the application of the FONLL formalism to the computation of DIS structure functions, working out explicit results up to order α_s. We use this example to highlight the main feature of the FONLL scheme, showing what is required in order to fully implement it, Ref. [1]. In section 3, the FONLL scheme applied to DIS is generalized in order to take into account intrinsic component of heavy quark c in the proton, Ref. [2], [5]. Here we depict the general lines which will then be followed in order to perform the analogue generalization for hadronic processes. In section 4, we describe how the standard FONLL method can be applied to an hadronic process, taking as example Higgs production in bottom-quark fusion, Ref. [6]. Explicit results for this specific process are reported, and we take them as starting point for our work.

In section 5, we discuss how the FONLL scheme applied to an hadronic process could be modified in order to take into account massive quarks in the initial state. We then specify the calculation to the case of Higgs production in bottom-quark fusion, working out suitable corrections which have to be added to the result of Ref. [6]. We give our results in terms of the specific partonic cross section characterizing the process. We point out the main general features of these outcomes and we compare them with those obtained in Ref. [2] and Ref. [5] in the contest of deep-inelastic scattering. Finally we give the full analytical expressions of our result, presenting outcomes for the calculations of the specific partonic cross sections we are interested in, and showing step by step how to compute them.

1 General formalism

In this section the basic formalism of massive and massless schemes is presented. The FONLL method is then described in its general ideas. In the next section it is then applied to DIS, using this concrete example to highlight its main features.

1.1 Massive and massless schemes

Perturbative processes in QCD involving heavy quarks can be treated within different factorization schemes. By "heavy" and "light" quarks we mean particles whose masses are respectively higher and lower than 1 GeV, such that only in the first case a perturbative treatment could be applicable. According to this definition the up, down and strange quarks are light, while the charm, bottom and top are heavy and can be described using perturbative theory. Precise results for computations of cross sections over a range of perturbative scales Q^2, require the incorporation of heavy quarks mass effects close to threshold $Q^2 \sim m^2$, and the resummation of collinear logarithms at higher scales $Q^2 \gg m^2$, where m is the mass of the heavy quark we are considering. This is achieved by using the two different schemes mentioned above.

In the first one the heavy quark is treated as a massless parton, there are no differences between heavy and light objects. Both light and heavy flavours participate in evolution QCD equations, through which collinear logarithms of $\frac{Q^2}{m^2}$, arising from heavy quarks emissions, are resummed to all orders. However, corrections suppressed by power of $\frac{m^2}{Q^2}$ are neglected.

On the other hand, in the second scheme the heavy quark is considered as a massive object which decouples from evolution equations and from the running of α_s, and the full dependence on m is retained order by order in perturbation theory. In this scheme only the light quarks PDFs evolve with standard DGLAP equations, as the factorization acts only on these flavours. The contributions coming from heavy quarks are evaluated at fixed order in perturbation theory, leaving explicitly unresummed collinear logarithms due to emissions of massive quarks.

These schemes are called respectively "massless" and "massive scheme", or alternatively "three" and "four flavour schemes", where the numbers refer to how many active flavours take part in the evolution equations. So, for example, thinking at quark charm as the massive one, we will talk of three and four flavour schemes, while we will have four and five flavour schemes when quark bottom is referred to as the heavy parton.

The point is that, dealing with massive partons, the quark mass acts as an IR regulator, so that radiative corrections involving massive quarks are finite, leading to logarithmic contributions of $\frac{Q^2}{m^2}$. Depending on the scale of the problem we are interested in, for massive quarks one may choose whether to factorize massive collinear logarithms (obtaining the massless scheme) or not (obtaining massive scheme). For massless quarks instead factorization is always necessary, since there isn't any mass acting as an IR regulator.

The massless scheme is more accurate for scale $Q^2 \gg m^2$ since here the collinear logarithms can become large, spoiling the perturbative convergence of the massive scheme. In this case, it is more appropriate to factorize and resum collinear logarithms associated to the heavy quark we are considering. Conversely, the massive scheme is more accurate close to threshold, where mass effects due to the production of the heavy quarks could become important. If a computation in the massive scheme is performed to high enough order in perturbation theory it will reproduce the results of the massless scheme, while the converse is not true, as in the massless scheme informations about mass effects due to heavy quarks are not encoded at all.

1.2 Combining fixed order and resummation

In the previous section we have introduced massless and massive scheme: while the first is accurate at high scales where corrections of $O\left(\frac{m^2}{Q^2}\right)$ are negligible, its use is not legitimate al lower scales near the heavy quark mass, where the second scheme is preferred. Therefore it is useful to combine these two schemes into one which works at all scales, reproducing the massive scheme results near the threshold and those of the massless scheme at higher scales.

In order to get a scheme which is valid to all scales, the power corrections of $O\left(\frac{m^2}{Q^2}\right)$ which are not included into the massless scheme, have to be taken into account, at a chosen fixed order: namely, it is sufficient to add to the massless scheme results the massive fixed order corrections which are encoded in the massive scheme. In this way we obtain a result retaining the accuracy of both massive and massless scheme: at a massive level we would have the fixed order precision corresponding to the number of orders that have been included in perturbation theory (FO, namely fixed order); on the other hand we would also have the resummation of large logarithms, keeping the accuracy of the starting massless scheme computation (NLL or subleading logarithms). This is the idea the FONLL method is based on.

In order to obtain such a result, it is sufficient to add massive and massless results and then subtract any double counted contributions. These correspond basically to constant terms and to those large logarithms which, because of heavy flavours factorization, are already resummed in the massless scheme but also explicitly appear in the massive one.

One of the main advantages of the FONLL method is that it involves only computations of physical quantities in well defined factorization schemes (massless or massive) and it permits to write down expressions at any order in perturbation theory in a straightforward way: first one evaluates the relevant massive diagrams for the process considered in the massive scheme, and then combine them linearly with the corresponding massless calculations. The double counted term can be obtained as the massless limit of the massive results and it corresponds to the fixed order expansion of the massless scheme.

To sum up, the FONLL method allows us to obtain results which are correct at high energy up to power suppressed terms (the power corrections of m^2/Q^2), and also in the threshold region up to subleading corrections (infact when $Q^2 \sim m^2$ the logarithms are note large), performing calculations in well defined factorization schemes.

In most applications FONLL was implemented first with the assumptions that heavy quarks were generated perturbatively, so without an intrinsic component of massive quarks in the initial state. In other words, according to this assumption in the massive scheme heavy quarks never appear in the initial states. Using this hypothesis, we will start showing explicit examples of usage of the FONLL prescription in calculation regarding DIS. Through this example, we highlight the main features of the method and we show what is required to its full implementation. Then we will present a generalization of the FONLL scheme which allows us to consider an intrinsic component of heavy quarks (the charm quark in the case of DIS).

In the second part of the thesis we will do the same for an hadronic process, showing the main differences between the latter and the DIS case.

2 Heavy quarks in deep-inelastic scattering

2.1 General structure

In this section we will see explicitly how the FONLL method works on the specific case of a generic deep-inelastic scattering structure function $F(x, Q^2)$. We will see later how this general formulation works for hadronic cross sections as well.

We assume a number n_l of light quarks q, with one massive quark h with mass m. The expression of $F(x, Q^2)$ in the massless scheme is given by

$$\begin{aligned} F^{(n_l+1)}\left(x, Q^2\right) &= x \int_x^1 \frac{dy}{y} \sum_{i=q,\bar{q},h,\bar{h},g} C_i^{(n_l+1)}\left(\frac{x}{y}, \alpha_s^{(n_l+1)}\left(Q^2\right)\right) f_i^{(n_l+1)}\left(y, Q^2\right) \\ &\equiv \sum_{i=q,\bar{q},h,\bar{h},g} C_i^{(n_l+1)}\left(\alpha_s^{(n_l+1)}\left(Q^2\right)\right) \otimes f_i^{(n_l+1)}\left(Q^2\right), \end{aligned} \tag{1}$$

where q is any light quark, h is the heavy quark and $n_l + 1$ is the total number of active flavours. This expression is accurate at large scale $Q^2 \gg m^2$, and using this scheme all collinear logarithms due to the massive quark are factorized into the definition of the PDFs and resummed through PDF evolution equations, just as for the light partons. Therefore the massive quark is included as light parton in the PDFs. These, together with $\alpha_s^{(n_l+1)}$, satisfy standard DGLAP evolution equations and renormalization group equation respectively, both with n_l+1 active flavours.

The expression of $F(x, Q^2)$ in the massive scheme is given by

$$\begin{aligned} F^{(n_l)}\left(x, Q^2\right) &= x \int_x^1 \frac{dy}{y} \sum_{i=q,\bar{q},g} C_i^{(n_l)}\left(\frac{x}{y}, \frac{Q^2}{m^2}, \alpha_s^{(n_l)}\left(Q^2\right)\right) f_i^{(n_1)}\left(y, Q^2\right) \\ &\equiv \sum_{i=q,\bar{q},g} C_i^{(n_l)}\left(\frac{Q^2}{m^2}, \alpha_s^{(n_l)}\left(Q^2\right)\right) \otimes f_i^{(n_1)}\left(Q^2\right), \end{aligned} \tag{2}$$

where now the coefficient functions C_i are calculated retaining the full mass dependence but contain unresummed logarithms of $\frac{Q^2}{m^2}$. This expression is accurate near the threshold $Q^2 \sim m^2$, and $f_i^{(n_l)}$ together with $\alpha_s^{(n_l)}$ obey respectively DGLAP and renormalization group equations with n_l active flavours.

The equations (2) and (1) are alternative expressions for the same structure function, written in terms of different elements i.e. α_s and PDFs in the two schemes. In order to carry on the FONLL method we have to express the massive scheme structure function in terms of $\alpha_s^{(n_l+1)}$ and $f_i^{(n_l+1)}$, therefore we need to know how the coupling constants and PDFs are related between the two schemes.

This relation is given by equations of the form

$$\alpha_s^{(n_l+1)}\left(Q^2\right) = \alpha_s^{(n_l)}\left(Q^2\right) + \sum_{i=2}^{+\infty} c_i\left(L\right) \times \left(\alpha_s^{(n_l)}\left(m^2\right)\right)^i \tag{3}$$

$$\begin{aligned} f_i^{(n_l+1)}\left(x, Q^2\right) &= \int_x^1 \frac{dy}{y} \sum_{j=q,\bar{q},g} K_{ij}\left(\frac{x}{y}, L, \alpha_s^{(n_l)}\left(Q^2\right)\right) f_j^{(n_l)}\left(y, Q^2\right) \\ &\equiv \sum_{j=q,\bar{q},g} K_{ij}\left(Q^2\right) \otimes f_j^{(n_l)}\left(Q^2\right), \end{aligned} \tag{4}$$

where

$$L \equiv \log\frac{Q^2}{m^2}. \tag{5}$$

The coefficients $c_i\left(L\right)$ are polynomials in L and can be obtained simply by the solution of the renormalization group equation for α_s. The functions K_{ij} can be expressed as an expansion in power of α_s, with coefficients that are polynomials in L. They are determined perturbatively, by requiring for example that the massive and massless results of equations (2) and (1) respectively are equal order by order in the same α_s. This computation is performed in Ref. [4]. Alternatively these matching coefficients can be computed as a matching between two effective theories of QCD, as done Ref. [8].

It is important to note how the index j of the coefficient K_{ij} only refers to a light quark or a gluon, not to the heavy flavour, as the heavy flavour PDFs in the massive scheme is considered to be zero: this is the assumption we made on the absence of an intrinsic heavy quark in the initial state. If we wanted to generalized the FONLL scheme in order to consider an initial fitted heavy quark PDF, as we will do later, we would have to consider additional coefficent functions, accounting for the presence of a non zero PDF for the heavy quark also in the massive scheme. For now, we consider only coefficients K_{ij} with i light or heavy and $j = g, q, \bar{q}$, which turn out to be non-zero only starting at $O\left(\alpha_s^2\right)$, order to which they have been fully computed in Ref. [4].

Therefore under the assumption that the heavy quark PDF is generated perturbatively, the first $2n_l + 1$ equations in Eq. (4) express the relation between the light quarks and gluon PDFs in the two schemes, while the last two give the heavy quark perturbative PDFs of the massless scheme in terms of the PDFs of the decoupled (massive) scheme.

In order to get an expression for the massive scheme in terms of $\alpha_s^{(n_l+1)}$ and $f^{(n_l+1)}$ one can invert equations (3) and (4) obtaining from Eq. (2)

$$F^{(n_l)}\left(x, Q^2\right) = x\int_x^1 \frac{dy}{y} \sum_{i=q,\bar{q},g} B_i\left(\frac{x}{y}, \frac{Q^2}{m^2}, \alpha_s^{(n_l+1)}\left(Q^2\right)\right) f_i^{(n_l+1)}. \tag{6}$$

Furthermore, using the DGLAP evolution equations in the absence of intrinsic heavy flavour, the heavy quark PDFs f_h, $f_{\bar{h}}$ of Eq. (1) can be written in terms of the light-quark PDFs $f_i^{(n_l)}$ with $i \neq h, \bar{h}$ at the scale m, convoluted with coefficient functions expressed as power series of $\alpha_s^{(n_l)}$, with coefficients that are polynomials in L, or, alternatively, in terms of the gluon and light-quark parton distribution $f_i^{(n_l+1)}$ at the scale Q^2 convoluted with coefficient function expressed as a power series in $\alpha_s^{(n_l+1)}(Q^2)$, with coefficients that are again polynomials in L. Thus the massless scheme expression Eq. (1) may be rewritten in terms of light quark PDFs only:

$$F^{(n_l+1)}\left(x, Q^2\right) = x \int_x^1 \frac{dy}{y} \sum_{i=q,\bar{q},g} A_i^{(n_l+1)}\left(\frac{x}{y}, L, \alpha_s^{(n_l+1)}\left(Q^2\right)\right) f_i^{(n_l+1)}\left(y, Q^2\right). \tag{7}$$

Expanding both $C_i^{(n_l+1)}$ coefficients and heavy quarks PDFs of Eq. (1), the general perturbative expression of the coefficients $A_i^{(n_l+1)}$ turns out to be:

$$A_i^{(n_l+1)}\left(z, L, \alpha_s^{(n_l+1)}\left(Q^2\right)\right) = \sum_{p=0}^{N} \left(\alpha_s^{(n_l+1)}\left(Q^2\right)\right)^p \sum_{k=0}^{\infty} A_i^{p,k}(z) \left(\alpha_s^{(n_l+1)}\left(Q^2\right) L\right)^k, \tag{8}$$

where, since this is the massless scheme, we notice the presence of the logarithmic terms $\alpha_s L$ resummed to all orders.

In order to match the two expressions of F, we expand also massive coefficient functions B_i of eq.(6):

$$B_i\left(z, \frac{Q^2}{m^2}, \alpha_s^{(n_l+1)}\left(Q^2\right)\right) = \sum_{p=0}^{P} \left(\frac{\alpha_s^{(n+1)}\left(Q^2\right)}{2\pi}\right)^p B_i^p\left(z, \frac{Q^2}{m^2}\right), \tag{9}$$

where P is the perturbative order we want to reach. We define the massless limit $B_i^{(0),p}\left(x, \frac{Q^2}{m^2}\right)$ of $B_i^p\left(x, \frac{Q^2}{m^2}\right)$, so that

$$\lim_{m\to 0}\left[B_i^p\left(x, \frac{Q^2}{m^2}\right) - B_i^{(0),p}\left(x, \frac{Q^2}{m^2}\right)\right] = 0. \tag{10}$$

As noticed before, the terms of eq.(9) which does not vanish when $Q^2 \gg m^2$ must also be present in the massless scheme. Therefore $B_i^{(0),p}\left(x, \frac{Q^2}{m^2}\right)$ must be equal to the $O\left(\alpha_s^p\right)$ factor of eq.(8)

$$B_i^{(0),p}\left(x, \frac{Q^2}{m^2}\right) = \sum_{k=0}^{p} A_i^{p-k,k}(x) L^k. \tag{11}$$

Therefore the terms contained in both massive and massless scheme can be obtained either as massless limit of the massive scheme or as a fixed order expansion in power of α_s of the massless scheme.

Giving the perturbative expansions above of both schemes, FONLL method can be implemented as follows: the contributions of Eq. (8) to coefficients $A_i^{(n_l+1)}$ of Eq. (7) which have a corresponding term $B_i^{(0),p}$ (if a certain contribution to coefficient A has a corresponding $B^{(0)}$ or not depends on the perturbative order to which coefficients B have been calculated) have to be replaced by the fully massive expression $B_i^p\left(x, \frac{Q^2}{m^2}\right)$ from Eq. (9). Thus the result obtained after this replacement includes all the mass suppressed effects that are not presented in Eq. (1) but are known from eq. (2).

In order to apply this procedure in a systematic way we define

$$F^{(n_l,0)}\left(x, Q^2\right) = x \int_x^1 \frac{dy}{y} \sum_{i=q,\bar{q},g} B_i^{(0)}\left(\frac{x}{y}, \frac{Q^2}{m^2}, \alpha_s^{(n_l+1)}\left(Q^2\right)\right) f_i^{(n_l+1)}\left(y, Q^2\right), \quad (12)$$

where

$$B_i^{(0)}\left(z, \frac{Q^2}{m^2}, \alpha_s^{(n_l+1)}\left(Q^2\right)\right) = \sum_{p=0}^{P} \left(\alpha_s^{(n_l+1)}\left(Q^2\right)\right)^p B_i^{(0),p}\left(z, \frac{Q^2}{m^2}\right), \quad (13)$$

where P stands for the order in $\alpha_s^{(n_l+1)}\left(Q^2\right)$ to which the massive scheme expression has been determined. Therefore $F^{(n_l,0)}\left(x, Q^2\right)$ contains the double counted terms mentioned before, contained both in the massive and massless scheme, which have to be subtracted from the final linear combination which gives FONLL construction. As we have just done, they can be obtained as the sum of all the contributions which do not vanish as the heavy quark mass tends to zero, namely constant and collinear logarithmic terms of the form $\log \frac{Q^2}{m^2}$, which are also contained in the massless expression as a consequence of the perturbative evolution of the PDFs. Alternatively they might be extracted from the massless result by expanding the massive quark PDFs in power of the strong coupling.

Therefore the FONLL expression is given by

$$\begin{aligned} F^{FONLL}\left(x, Q^2\right) &= F^{(n_l+1)}\left(x, Q^2\right) + F^{(n_l)}\left(x, Q^2\right) - F^{(n_l,0)}\left(x, Q^2\right) \\ &= F^{(d)}\left(x, Q^2\right) + F^{(n_l)}\left(x, Q^2\right), \end{aligned} \quad (14)$$

with

$$F^{(d)}\left(x, Q^2\right) = F^{(n_l+1)}\left(x, Q^2\right) - F^{(n_l,0)}\left(x, Q^2\right). \quad (15)$$

We notice that, thanks to Eq. (10), when $Q^2 \gg m^2$ the FONLL expression reproduced the massless scheme results, while for $Q^2 \sim m^2$ it differs from the massive scheme through the difference term $F^{(d)}$ which is subleading in α_s.

A final note regards the definition of perturbative ordering in FONLL: in order to ensure that Eq. (15) works properly, we have to calculate $F^{(n_l+1)}$ at order in $\alpha_s^{(n_l+1)}$ which il at least as high as that of $F^{(n_l)}$, so that in Eq. (15) we are subtracting terms which are actually present in $F^{(n_l+1)}$. However we may also compute $F^{(n_l)}$ with higher precision, and define $F^{(n_l,0)}$ retaining only those terms which are also present in $F^{(n_l+1)}$. Making this choice it would not be true that for $Q^2 \gg m^2$ massless scheme results are reproduced, because $F^{(n_l)}$ and $F^{(n_l,0)}$ no longer cancel in this limit since in the latter some terms have been excluded. However is still true that when $Q^2 \gg m^2$ FONLL reduces to massless results up to mass suppressed terms and terms of higher order in α_s, coming from this mismatch in accuracy. To sum up, an advantage of the FONLL method is that the perturbative order at which heavy quark terms are included in $F^{(n_l)}$ and $F^{(n_l+1)}$ (fixed in the first case, logarithmically resummed in the second) can be chosen freely. Keeping into account for this observation, we will consider in particular three options for perturbative ordering in the FONLL prescription. Thinking about DIS we have:

- FONLL-A: contributions to $F^{(n_l)}$ from heavy quark are computed at LO, which is $O(\alpha_s)$ in DIS, while those to $F^{(n_l+1)}$ are computed at NLL which is also $O(\alpha_s)$. There is no mismatch in accuracy between results of the two schemes.

- FONLL-B: contributions to $F^{(n_l)}$ from heavy quark are computed at NLO, which is $O(\alpha_s^2)$ in DIS, while those to $F^{(n_l+1)}$ are still computed at NLL which is $O(\alpha_s)$. In this case the massive expression exceeds in accuracy the massless one when L is not large. Thus, as mentioned above, we define $F^{(n_l,0)}$ retaining only those terms which are also present in $F^{(n_l+1)}$.

- FONLL-C: contributions to $F^{(n_l)}$ from heavy quark are computed at NLO, which is $O(\alpha_s^2)$ in DIS, while those to $F^{(n_l+1)}$ are computed at NNLL which is also $O(\alpha_s^2)$. There is no mismatch in accuracy between results of the two schemes.

Throughout the whole thesis we will use the notation N^kLL to refer to the resummation of collinear logs of the heavy quark mass, namely by LL we mean a computation in which $\left(\alpha_s \log \frac{m^2}{Q^2}\right)$ is treated as order one. In other words, with LL, NLL etc. we refer to the order at which DGLAP equations are solved, while with LO, NLO, etc. we refer to the order at which hard massive cross section are computed in perturbation theory. By "leading order" we always mean the lowest nontrivial order at which the process starts occurring. Therefore the generalization to hadronic processes will require a relabeling of perturbative orders, since the order at which massive scheme start being non-zero is process dependent.

In conclusion, in this subsection we have introduced the general formalism concerning FONLL method, using as example the specific case of DIS structure functions. Finally we have discussed and named some possible perturbative orders to which FONLL method could be implemented.

2.2 Implementation of the FONLL method

In this subsection we will work out explicit FONLL results for structure functions in DIS up to $O(\alpha_s)$. This is useful because of a variety of reasons: it allows to see clearly how the FONLL prescription works, how matching conditions are obtained at a generic scale, and to find out explicitly the pieces of the massive scheme already contained in the massless one, giving a clear example of what exactly happens. Furthermore in the next sections of this work we will follow the general lines described in this subsection to work out the generalizations required for hadronic processes.

Firstly, a detailed discussion of the change between scheme with n_l flavours and that with $n_l + 1$ flavours is required. This is the key point on which all the next generalizations of the FONLL scheme will be based on.

We start writing explicit expression for Eq. (3), using the solution of the renormalization group equation for α_s which in general reads

$$\begin{aligned} \alpha_s\left(Q^2\right) &= \alpha_s\left(m^2\right) - b_0 \log\frac{Q^2}{m^2}\alpha_s^2\left(m^2\right) + O\left(\alpha_s^3\right) \\ b_0 &= \frac{33-2n_f}{12\pi}. \end{aligned} \tag{16}$$

Therefore we have

$$\begin{aligned} \alpha_s^{(n_l+1)}\left(Q^2\right) &= \alpha_s\left(m^2\right) - \frac{33-2\left(n_l+1\right)}{12\pi}\log\frac{Q^2}{m^2}\alpha_s^2\left(m^2\right) + O\left(\alpha_s^3\right) \\ &= \alpha_s\left(m^2\right) - \frac{33-2n_l}{12\pi}\log\frac{Q^2}{m^2}\alpha_s^2\left(m^2\right) + \frac{1}{6\pi}\log\frac{Q^2}{m^2}\alpha_s^2\left(m^2\right) + O\left(\alpha_s^3\right) \\ &= \alpha_s^{(n_l)}\left(Q^2\right) + \frac{2T_R}{3}\frac{L}{2\pi}\alpha_s^2\left(m^2\right) + O\left(\alpha_s^3\right), \end{aligned} \tag{17}$$

from which we read that at scale $Q^2 = m^2$ the couplings in the two schemes differ by terms of order $O\left(\alpha_s^3\right)$

$$\alpha_s^{(n_l+1)}\left(m^2\right) = \alpha_s^{(n_l)}\left(m^2\right) + O\left(\alpha_s^3\right). \tag{18}$$

As for matching PDFs in the two schemes, we proceed in two stages: first we match the two schemes at a scale $Q^2 \sim m^2$ then we evolve to a generic scale Q^2, using DGLAP evolution equations. In the absence of intrinsic heavy quark contributions, the matching conditions at scale m^2 are

$$
\begin{aligned}
f_i^{(n_l+1)}\left(x,m^2\right) &= \sum_{j=q,\bar{q},g} K_{ij}\left(m^2\right)\otimes f_j^{(n_l)}\left(m^2\right) \\
&= \int_x^1 \frac{dz}{z}\sum_{j=q,\bar{q},g} K_{ij}\left(z,m^2\right) f_j^{(n_l)}\left(\frac{x}{z},m^2\right) \\
&= f_i^{(n_l)}\left(x,m^2\right) + \frac{\alpha_s^2}{4\pi^2}\int \frac{dz}{z}\sum_j K_{ij}^{(2)}\left(z,m^2\right) f_j^{(n_l)}\left(\frac{x}{z},m^2\right) + O\left(\alpha_s^3\right)
\end{aligned}
\tag{19}
$$

where $i,j=q,\bar{q},g$. Evolving this expression to scale Q^2 we would return to the general expression of Eq. (4). From the second to the third line of Eq. (19) the coefficient K_{ij} has been expanded as a power series of α_s. Since the index i and j are referred only to light flavour, as mentioned in previous sections K_{ij} becomes nontrivial starting at $O\left(\alpha_s^2\right)$, and the function $K_{qq}^{(2)}(z)$, $K_{gq}^{(2)}(z)$ and $K_{gg}^{(2)}(z)$ have been fully computed in Ref. [4]. As for heavy quarks instead we have

$$
f_h^{(n_l+1)}\left(x,m^2\right) = f_{\bar{h}}^{(n_l+1)}\left(x,m^2\right) = O\left(\alpha_s^2\right), \tag{20}
$$

as $K_{hi}^{(2)}(z)$ with $i=q,\bar{q},g$ start at order $O\left(\alpha_s^2\right)$.

As seen above, in order to implement the FONLL method we need to express the massive scheme PDFs in terms of the massless scheme ones at a generic scale Q^2. In order to obtain explicit results for the FONLL scheme up to order $O\left(\alpha_s\right)$, matching condition at order $O\left(\alpha_s\right)$ are enough. Therefore we evolve both $f_i^{(n_l+1)}$ and $f_i^{(n_l)}$ in the respective schemes with n_l+1 and n_l active flavours, solving the DGLAP equations. Thus from Eq. (19) evolving up to order $O\left(\alpha_s\right)$ we have

$$
\begin{aligned}
f_i^{(n_l+1)}\left(x,Q^2\right) &= f_i^{(n_l)}\left(x,Q^2\right) \\
&+\alpha_s L\int_x^1 \frac{dz}{z}\sum_j\left[P_{ij}^{(n_l+1),0}(z) - P_{ij}^{(n_l),0}(z)\right] f_j^{(n_l)}\left(\frac{x}{y},Q^2\right) + O\left(\alpha_s^2\right),
\end{aligned}
\tag{21}
$$

where P_{ij}^0 are the leading order Altarelli-Parisi splitting functions in the two schemes, and the sum runs over all light flavour (we are using the same definition of splitting functions used in Ref. [11], but with an additional factor equal to $\frac{1}{2\pi}$). Using

$$
P_{ij}^{(n_l+1),0}(z) - P_{ij}^{(n_l),0}(z) = -\delta_{ig}\delta_{jg}\frac{2T_R}{3\left(2\pi\right)}\delta\left(1-z\right), \tag{22}
$$

it follows that

$$
\begin{aligned}
f_g^{(n_l+1)}\left(x,Q^2\right) &= f_g^{(n_l)}\left(x,Q^2\right) - \alpha_s\frac{L}{2\pi}\frac{2T_R}{3} f_g^{(n_l)}\left(x,Q^2\right) + O\left(\alpha_s^2\right) \\
f_q^{(n_l+1)}\left(x,Q^2\right) &= f_q^{(n_l)}\left(x,Q^2\right) + O\left(\alpha_s^2\right)
\end{aligned}
\tag{23}
$$

These are explicit expressions for the matching conditions connecting massless and massive scheme PDFs up to order $O(\alpha_s)$.

Now we have all the ingredients required to write explicitly FONLL results for structure functions of DIS at order α_s. In order to simplify the computation, it is convenient to separate off the light and heavy contributions to a generic DIS structure function and to its corresponding coefficient function. Thus we write

$$\begin{aligned} F\left(x,Q^2\right) &= F_l\left(x,Q^2\right) + F_h\left(x,Q^2\right) \\ C_i\left(x,\alpha_s\left(Q^2\right)\right) &= C_{i,l}\left(x,\alpha_s\left(Q^2\right)\right) + C_{i,h}\left(x,\alpha_s\left(Q^2\right)\right), \end{aligned} \tag{24}$$

where F_h and F_l define respectively the contribution to $F\left(x,Q^2\right)$ which survives if only the electric charge of the heavy quark is non-zero, or that which survives if the electric charge of the heavy quark vanishes. In other words, the labels h and l stand for the quark to which the virtual photon couples, while i denotes the parton which enters the hard scattering process. In the following example we will use results concerning coefficient functions which are true in the case of the structure function $F_2\left(x,Q^2\right)$ in electromagnetic deep-inelastic scattering.

Equation (19) means that at scale $Q^2 \sim m^2$, at order $O(\alpha_s)$ the gluon and light quarks PDFs in the massive and massless schemes have to be equal, while Eq. (20) tells that the non-intrinsic heavy quark PDFs should be taken to vanish. The matching conditions at a generic scale Eq. (23) are used to determine the coefficient function B_i in terms of the original massive ones $C_i^{(n_l)}$. Light quark coefficient functions start at $O\left(\alpha_s^0\right)$, while the gluon ones at $O\left(\alpha_s^1\right)$ therefore, since the correction at order $O(\alpha_s)$ is non zero only for the gluon PDF, it follows that coefficient functions B_i start being different from C_i only at $O\left(\alpha_s^2\right)$:

$$B_i\left(z,\frac{Q^2}{m^2},\alpha_s^{(n_l+1)}\left(Q^2\right)\right) = C_i^{(n_l)}\left(z,\frac{Q^2}{m^2},\alpha_s^{(n_l+1)}\left(Q^2\right)\right) + O\left(\alpha_s^2\right). \tag{25}$$

Furthermore at this order all light coefficient functions $C_{i,l}$ with $i = q,\bar{q},g$ are the same in the massive and massless scheme

$$C_{i,l}^{(n_l)}\left(z,\frac{Q^2}{m^2},\alpha_s^{(n_l+1)}\left(Q^2\right)\right) = C_{i,l}^{(n_l+1)}\left(z,\frac{Q^2}{m^2},\alpha_s^{(n_l+1)}\left(Q^2\right)\right) + O\left(\alpha_s^2\right), \tag{26}$$

since in order to have some differences we should consider diagrams of higher orders. Therefore we have:

$$\begin{aligned} F_l^{FONLL}\left(x,Q^2\right) &= F_l^{(d)}\left(x,Q^2\right) + F_l^{(n_l)}\left(x,Q^2\right) \\ &= x\sum_{i=h,\bar{h}}\int_x^1 \frac{dy}{y}\,C_{i,l}^{(n_l+1)}\left(\frac{x}{y},\alpha_s\left(Q^2\right)\right) f_i^{(n_l+1)}\left(y,Q^2\right) \\ &+ x\sum_{i\neq h,\bar{h}}\int_x^1 \frac{dy}{y}\,C_{i,l}^{(n_l+1)}\left(\frac{x}{y},\alpha_s\left(Q^2\right)\right) f_i^{(n_l+1)}\left(y,Q^2\right) + O\left(\alpha_s^2\right) \\ &= F_l^{(n_l+1)}\left(x,Q^2\right) + O\left(\alpha_s^2\right), \end{aligned} \tag{27}$$

i.e. the FONLL expression of F_l reduces to the massless scheme one.

As for heavy coefficient function $C_{i,h}$ for i corresponding to any light quark, they also are the same in massless and massive schemes at $O(\alpha_s)$, since in both schemes they vanish at this order. However, accounting for the first diagram in Fig. (1), the gluon coefficient function is different

$$C_{g,h}^{(n_l)}\left(z, \frac{Q^2}{m^2}, \alpha_s\left(Q^2\right)\right) = \frac{\alpha_s\left(Q^2\right)}{2\pi} C_g^{(n_l),1}\left(z, \frac{Q^2}{m^2}\right) + O\left(\alpha_s^2\right), \tag{28}$$

where

$$\begin{aligned} &C_g^{(n_l),1}\left(z, \frac{Q^2}{m^2}\right) = \theta\left(W^2 - 4m^2\right) \times \\ &T_R[\left(z^2 + (1-z)^2 + 4\epsilon z(1-3z) - 8\epsilon^2 z^2\right)\log\frac{1+v}{1-v} \\ &+ (8z(1-z) - 1 - 4\epsilon z(1-z))\, v], \end{aligned} \tag{29}$$

with

$$\epsilon \equiv \frac{m^2}{Q^2}; \qquad v \equiv \sqrt{1 - \frac{4m^2}{W^2}}; \qquad W \equiv \sqrt{\frac{Q^2(1-x)}{x}}; \tag{30}$$

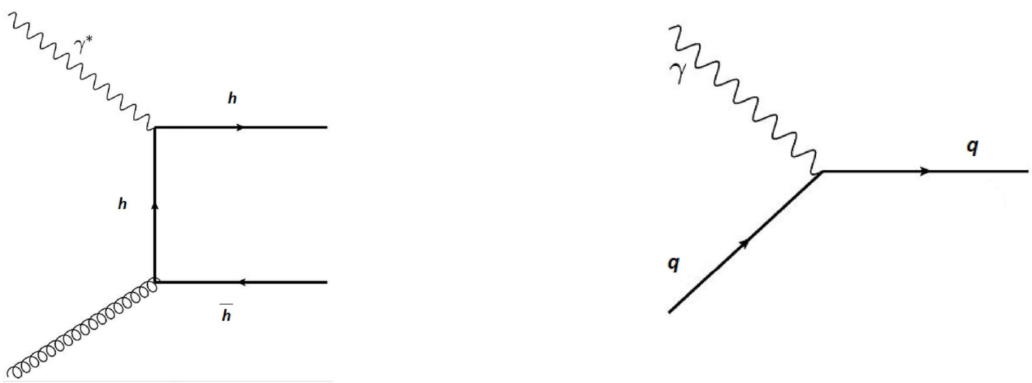

Figure 1: From left to right, rapresentative Feynman diagrams for LO contribution to the structure function in 4FS and 3FS

The coupling $\alpha_s\left(Q^2\right)$ in these expressions can equivalently be either $\alpha_s^{(n_l+1)}$ or $\alpha_s^{(n_l)}$, since they differ by terms of order $O\left(\alpha_s^2\right)$. This is the explicit expression for the coefficient function obtained from the massive scheme, which is accurate when $Q^2 \sim m^2$ i.e. when $W^2 \sim 4m^2$. In this range of scales Q^2, the logarithmic contribution contained in Eq. (29) is still finite. The massless limit of Eq. (29) is

$$\begin{aligned} &B_{g,h}^{(0),1}\left(z, \frac{Q^2}{m^2}\right) = 2e_h^2 C_g^{(n_l,0),1}\left(z, \frac{Q^2}{m^2}\right) \\ &C_g^{(n_l,0),1}\left(z, \frac{Q^2}{m^2}\right) = T_R\left[\left(z^2 + (1-z)^2\right)\log\frac{Q^2(1-z)}{m^2 z} + (8z(1-z) - 1)\right]. \end{aligned} \tag{31}$$

Here we see explicitly the terms contained in both massive and massless scheme, which have to be subtracted to avoid being double counted. The logarithmic term of Eq. (31) is resummed at all orders in the massless scheme, while here it appears explicitly. In the limit $Q^2 = m^2$ the massless scheme coefficient function is reproduced

$$C_{g,h}^{(n_l+1)}\left(z,\alpha_s\left(Q^2\right)\right) = \frac{\alpha_s\left(Q^2\right)}{2\pi} 2e_h^2 C_g^{(n_l,0),1}\left(z,1\right). \tag{32}$$

Therefore the FONLL result for the heavy component is

$$F_h^{FONLL}\left(x,Q^2\right) = F_h^{(n_l)}\left(x,Q^2\right) + F_h^{(d)}\left(x,Q^2\right), \tag{33}$$

with the contributions on the right hand side given by

$$F_h^{(n_l)}\left(x,Q^2\right) = x\int_x^1 \frac{dy}{y} C_{g,h}^{(n_l)}\left(\frac{x}{y},\frac{Q^2}{m^2},\alpha_s\left(Q^2\right)\right) f_g^{(n_l+1)}\left(y,Q^2\right), \tag{34}$$

and

$$F_h^{(d)}\left(x,Q^2\right) = x\int_x^1 \frac{dy}{y}\left\{C_{h,h}^{(n_l+1)}\left(\frac{x}{y},\alpha_s\left(Q^2\right)\right)\left[f_h^{(n_l+1)}\left(y,Q^2\right) + f_{\bar{h}}^{(n_l+1)}\left(y,Q^2\right)\right]\right.$$
$$\left. + \left(C_{g,h}^{(n_l+1)}\left(\frac{x}{y},\alpha_s\left(Q^2\right)\right) - B_{g,h}^{(0)}\left(\frac{x}{y},\frac{Q^2}{m^2}\right)\right) f_g^{(n_l+1)}\left(y,Q^2\right)\right\}. \tag{35}$$

Using the leading order results of QCD evolution equations

$$\begin{aligned} f_h\left(y,Q^2\right) &= f_{\bar{h}}\left(y,Q^2\right) \\ &= \frac{\alpha_s\left(Q^2\right)}{2\pi} L\int \frac{dz}{z} T_R\left(z^2 + (1-z)^2\right) f_g\left(\frac{x}{y},Q^2\right) + O\left(\alpha_s^2\right) \\ &= \alpha_s\left(Q^2\right) L P_{qg}^{(0)} \otimes f_g\left(Q^2\right), \end{aligned} \tag{36}$$

and noting that

$$C_{h,h}^{(n_l)}\left(x,\alpha_s\left(Q^2\right)\right) = \delta\left(1-x\right) + O\left(\alpha_s\right), \tag{37}$$

this result can be checked in its entirety replacing Eq. (31), (32), (36) and (37) in Eq. (35), verifying explicitly the difference term cancel up to order $O\left(\alpha_s^2\right)$, as it has to happen. Thus in the region where L is not large (i.e. in the region where eq.(36) makes sense) the FONLL expression at $O\left(\alpha_s\right)$ for the heavy contribution coincides with the one in the massive scheme. Note that in this way we have explicitly checked that the double counted terms can be obtained in two ways as expressed by Eq. (11): either as the massless limit of the massive scheme or as the fixed order expansion in power of α_s of the massless scheme. Namely we have verify that

$$\begin{aligned} & B^{(0)}_{g,h}\left(\frac{Q^2}{m^2}\right)\otimes f_g^{(n_l+1)}\left(Q^2\right)=\alpha_s\left(Q^2\right)B^{(0),1}_{g,h}\left(\frac{Q^2}{m^2}\right)\otimes f_g^{(n_l+1)}\left(Q^2\right)+O\left(\alpha_s^2\right) \\ & =\lim_{Q^2\to m^2}\left[\sum_{i=h,\bar{h}}C^{(n_l+1)}_{i,h}\otimes f_i^{(n_l+1)}\left(Q^2\right)+C^{(n_l+1)}_{g,h}\otimes f_g^{(n_l+1)}\left(Q^2\right)\right] \\ & =\lim_{Q^2\to m^2}\left[\sum_{i=h,\bar{h}}\left(C^{(n_l+1),0}_{i,h}+\alpha_s\left(Q^2\right)C^{(n_l+1),1}_{i,h}\right)\otimes f_i^{(n_l+1)}\left(Q^2\right)\right. \\ & \qquad\qquad\left.+\alpha_s\left(Q^2\right)C^{(n_l+1),1}_{g,h}\otimes f_g^{(n_l+1)}\left(Q^2\right)\right] \\ & =\alpha_s\left(Q^2\right)\left[\sum_{i=h,\bar{h}}C^{(n_l+1),0}_{i,h}\otimes LP^{(0)}_{qg}+C^{(n_l+1),1}_{g,h}\right]\otimes f_g^{(n_l+1)}\left(Q^2\right)+O\left(\alpha_s^2\right), \end{aligned} \tag{38}$$

where in the last line we have used eq.(36).

Therefore we can re-write eq.(33) as

$$\begin{aligned} F_h^{FONLL}\left(x,Q^2\right) & =\sum_{i=h,\bar{h}}\left(C^{(n_l+1),0}_{i,h}+\alpha_s\left(Q^2\right)C^{(n_l+1),1}_{i,h}\right)\otimes f_i^{(n_l+1)}\left(Q^2\right) \\ & +\alpha_s\left(Q^2\right)C^{(n_l+1),1}_{g,h}\otimes f_g^{(n_l+1)}\left(Q^2\right) \\ & +\alpha_s\left(Q^2\right)\left(C^{(n_l),1}_{g,h}\left(\frac{Q^2}{m^2}\right)-\sum_{i=h,\bar{h}}C^{(n_l+1),0}_{i,h}\otimes LP^{(0)}_{qg}-C^{(n_l+1),1}_{g,h}\right) \\ & \otimes f_g^{(n_l+1)}\left(Q^2\right)+O\left(\alpha_s^2\right) \\ & =\sum_{i=h,\bar{h}}\left(C^{(n_l+1),0}_{i,h}+\alpha_s\left(Q^2\right)C^{(n_l+1),1}_{i,h}\right)\otimes f_i^{(n_l+1)}\left(Q^2\right) \\ & +\alpha_s\left(Q^2\right)\left(C^{(n_l),1}_{g,h}\left(\frac{Q^2}{m^2}\right)-\sum_{i=h,\bar{h}}C^{(n_l+1),0}_{i,h}\otimes LP^{(0)}_{qg}\right) \\ & \otimes f_g^{(n_l+1)}\left(Q^2\right)+O\left(\alpha_s^2\right). \end{aligned} \tag{39}$$

In the next section we will use the results shown till now in the form of this last equation. To sum up, in this section we have presented explicit result for DIS structure function up to order $O\left(\alpha_s\right)$, pointing out the main features of the computation and showing the difference between massive and massless coefficient. We have noticed how, in the massive scheme result, collinear logarithms due to heavy quark emissions explicitly appear, which are instead resummed to all orders in massless computations. In the next section we will describe how this approach can be generalized in order to account for a fitted heavy quark in the initial state.

3 Intrinsic heavy quark in deep-inelastic scattering

In this section we will see how FONLL method previously introduced can be generalized accounting for an intrinsic component of heavy quark in the initial state. We depict the basic ideas of such a generalization and give results up to $O(\alpha_s)$ for structure function in DIS, first obtained in Ref. [5]. We point out the main lines which will be followed in doing the same generalization in an hadronic process.

3.1 Motivations

The construction of FONLL method introduced in previous sections is made under the assumption that the heavy quark content of colliding hadrons is generated perturbatively, namely heavy quarks are generated from radiation by light partons and are not originally contained in the initial state. This assumption could be a limitation and possibly a source of bias. The reasons are both of principle and practice. First the heavy quark PDF might have a non vanishing intrinsic origin, such that it does not vanish at scale lower than the threshold one. This could be remarkable especially in the case of the charm quark, because of arguments suggesting that the proton may contain a significant charm component. Secondly, even if the heavy quark was generated perturbatively, and thus its PDF vanishes below the physical threshold for production, it is not clear what the values of this threshold should be. For sure it is a scale of non-perturbative origin, so we don't know its precise value: it could be the heavy quark mass, but it may be also the mass of some bound state. Therefore we would have results depending on some arbitrary starting scale $Q_0 \sim m$, not better defined, at which the perturbative boundary conditions are imposed. It is the scale at which the heavy quark is perturbative, and it really is an ambiguity, since it remains in all the final results. Finally, when we match the massive and the massless schemes through the matching conditions, we have to choose the scale where to perform the initial matching. Depending on this choice, in the matching coefficient K_{ij} different logarithmic terms will appear, which could indeed affect our final results at low order. This dependence is due to the fact that the matching between massive and massless scheme is performed at finite order. Despite in a matched calculation it would disappear at high enough perturbative orders, at low order it could be non negligible. These three problems can be solved by introducing a fitted heavy quark PDF, which could describe a possible intrinsic component of the PDF, and also reabsorb in the initial condition the dependence on the non-perturbative starting scale. In this way heavy quark PDFs would be parametrized and determined along with gluon and light quark PDFs through an appropriate fit. Thus the distinction between perturbative generated and intrinsic component would become irrelevant, and whether or not the heavy quark PDF vanishes and at what scale would be answered by the fit. From now on, we will refer to the initial-state heavy quark in the massive scheme as intrinsic heavy quark, without worrying if it is of perturbative or intrinsic origin.

3.2 General structure

In this subsection we will show how FONLL method of previous section can be generalized in order to account for a fitted heavy quark PDF, for the specific example of structure function in DIS. Throughout this section we will refer to the massive scheme as three flavor scheme (3FS) and to the massless scheme as four flavor scheme (4FS) as already mentioned before, while we will refer to charm quark as the heavy parton.

Both 3FS and 4FS have to be changed. As for 4FS, in absence of an intrinsic heavy quark, the massless scheme PDFs are completely determined by perturbative evolution from a vanishing boundary condition imposed at a scale of the order of the charm mass. Considering this scale to be exactly equal to the charm mass, we should have $f_h^{(4)}(x, m^2) = f_{\bar{h}}^{(4)}(x, m^2) = 0$ while $f_h^{(4)}(x, Q^2)$ and $f_{\bar{h}}^{(4)}(x, Q^2)$ for $Q^2 > m^2$ satisfy perturbative evolution with 4 active flavours. If we introduced a fitted heavy quark PDF we would simply have to relax the vanishing boundary condition, imposing instead that $f_h^{(4)}(x, m^2)$ and $f_{\bar{h}}^{(4)}(x, m^2)$ are given by some parametrization obtained by a proper fit. Apart from changing the boundary conditions, the 4FS remains unchanged.

As regards 3FS, we have to account for the presence of charm quark at all scales, therefore also when $Q^2 < m^2$ there are non vanishing heavy quark PDFs $f_h^{(3)}(x, m^2)$, $f_{\bar{h}}^{(3)}(x, m^2)$. It is important to notice that, since in this scheme the heavy quark is treated as a massive object which decouples from QCD evolution equations, these PDFs are scale independent. Thus in the expression of the 3FS we have to introduce new contributions to the structure functions, accounting for heavy quarks in the initial state.

Decomposing the structure function into an heavy and light component as done in Eq. (24), the heavy component F_h now receive a new contribution from $f_h^{(3)}$ and $f_{\bar{h}}^{(3)}$ which start at the parton model level, namely at $O(\alpha_s^0)$ whose representative Feynman diagrams are given in Fig. (2). The light structure functions F_l instead receive new contributions starting at order $O(\alpha_s^2)$.

Thus we define a correction term ΔF which has to be added to the previous

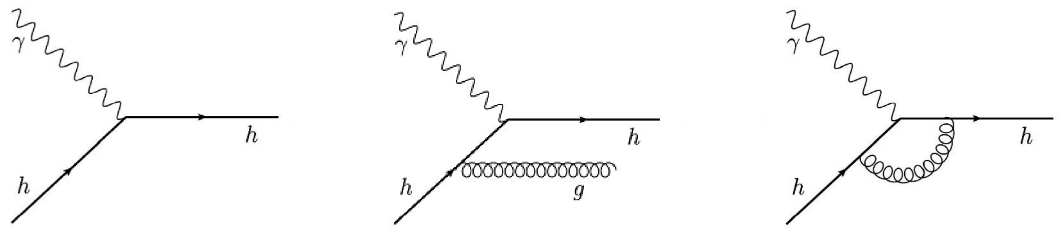

Figure 2: Feynman diagrams for the new contributions to $F_h\left(x,Q^2\right)$ induced by the presence of the heavy quark in the initial state. The fermion line represnts the heavy quark, and from left to right LO and NLO real and virtual corrections are shown.

expression of FONLL results in order to take into account the presence of an intrinsic heavy quark component. As the 4FS expression does not change (apart from the boundary conditions) in the correction term only the new contribution raising from the 3FS appears. Therefore, as for the correction to the heavy component of the structure function we have, up to $O\left(\alpha_s\right)$:

$$\Delta F_h\left(x,Q^2\right)=\sum_{i=h,\bar{h}}\left[C_{i,h}^{(3)}\left(\frac{Q^2}{m^2},\alpha_s^{(3)}\left(Q^2\right)\right)-C_{i,h}^{(3,0)}\left(\frac{Q^2}{m^2},\alpha_s^{(3)}\left(Q^2\right)\right)\right]\otimes f_i^{(3)}, \tag{40}$$

where, as explained above, $f_h^{(3)}$ and $f_{\bar{h}}^{(3)}$ are scale independent (from now on, we will omit the "h" index of $C_{i,h}^{(3)}$ and $C_{i,h}^{(3)}$, since in what follows we will refer only to the heavy component of the coefficient functions). Thus the FONLL expression of the previous section, denoted F^{FLNR} in the following equation, up to order $O\left(\alpha_s\right)$ is generalized to

$$F^{FONLL}\left(x,Q^2\right)=F^{FLNR}\left(x,Q^2\right)+\Delta F_h\left(x,Q^2\right). \tag{41}$$

Thus we have presented the general ideas the generalization of FONLL method is based on. In the next section we will see how to implement it.

3.3 Implementation

In this subsection we report the main steps through which the FONLL method is explicitly generalized to take account of a fitted heavy quark PDF, in the example of a generic DIS structure function, up to order α_s.

Starting from Eq. (40), the FONLL expression is obtained by expressing the 3FS coupling and PDFs in terms of the massless ones, through suitable matching conditions. As already discussed, this is done first by a matching at some fixed scale, for example at a scale $Q^2 = m^2$ and then evolving both the 3FS and 4FS couplings and PDFs in their respective schemes with 3 and 4 active flavours.

While Eq. (17) still holds, Eq. (19) and (20) have now to be generalized, to keep into account of the presence of new massive PDFs $f_h^{(3)}$ and $f_{\bar{h}}^{(3)}$. In Eq. (19) the matching coefficients $K_{ij}\left(m^2\right) = \sum_n \alpha_s^n K_{ij}^{(n)}\left(m^2\right)$ start receiving non-zero contributions at $O\left(\alpha_s^2\right)$, namely for $i,j = q,\bar{q},g$ we have $K_{ij}^{(0)} = \delta_{ij}$, while the $K_{hi}\left(m^2\right)$ functions with $i = q,\bar{q},g$ start at $O\left(\alpha_s^2\right)$, as seen in Eq. (20).

The coefficient $K_{ih}\left(m^2\right)$, which in the previous case where irrelevant, have now to be considered, due to the presence of heavy quark PDFs also in the massive scheme. They can be calculated up to $O\left(\alpha_s\right)$ in the same way of the other matching coefficients, using the known $O\left(\alpha_s\right)$ expression of the charm-initiated massive coefficient functions of DIS, and they are first addresses in Ref. [5]. Their explicit expressions at a generic scale Q^2 are given in the equation below:

$$
\begin{aligned}
K_{hh}\left(Q^2\right) &= K_{\bar{h}\bar{h}}\left(Q^2\right) \\
&= 1 + \alpha_s \left[\bar{P}_{qq}^{(0)}\left(z\right)\left(\log\frac{Q^2}{m^2\left(1-z\right)^2} - 1\right)\right]_+ + O\left(\alpha_s^2\right)
\end{aligned} \tag{42}
$$

$$
K_{gh}\left(Q^2\right) = K_{g\bar{h}}\left(Q^2\right) = \alpha_s P_{gq}^{(0)}\left(\log\frac{Q^2}{m^2 z^2} - 1\right) + O\left(\alpha_s^2\right),
$$

with

$$
\bar{P}_{qq}^{(0)}\left(z\right) = \frac{C_F}{2\pi}\frac{1+z^2}{1-z}, \quad P_{gq}^{(0)} = \frac{C_F}{2\pi}\frac{1+\left(1-z\right)^2}{z}. \tag{43}
$$

Thus we notice that $K_{hh}\left(m^2\right)$ already receives non-trivial contributions to order $O\left(\alpha_s\right)$, $K_{gh}\left(m^2\right)$ starts at order $O\left(\alpha_s\right)$ and finally $K_{qh}\left(m^2\right)$, not reported here, starts at higher orders. Their explicit expression was originally obtained considering the specific case of the DIS structure functions, but since they are universal quantities which connect two different factorization schemes, they can be used in computing whatever process requires initial-state heavy quarks. Furthermore there is no dependence from the number of light quarks, which allows us to use these results for connecting any $n_l + 1$ flavour scheme to its corresponding massive n_l flavour scheme. Therefore we will use these same matching coefficients in performing calculations for hadronic processes, with factorization schemes characterized by five and four active flavours.

Thus matching conditions given in Eq. (19) and (20) have to be generalized and re-expressed as

$$
f_i^{(n_l+1)}\left(m^2\right) = \sum_j K_{ij}\left(m^2\right) \otimes f_j^{(n_l)}\left(m^2\right), \tag{44}
$$

with $i,j = q,\bar{q},g,h,\bar{h}$. Unlike the previous case they are already non-trivial at $O(\alpha_s)$ due to the new components $K_{ih}(m^2)$ of the matching functions given in eq.(42).

As already done before, Eq. (44) have to be evolved using both the massive and massless schemes, in order to get matching condition at a generic scale Q^2. In particular, remembering that the 3FS heavy quark PDFs are scale independent we can write the matching conditions at a generic scale Q^2 as

$$\begin{aligned} f_h^{(3)} =& f_h^{(4)}\left(Q^2\right) - \alpha_s^{(3)}\left(Q^2\right) K_{bb}^{(1)}\left(m^2\right) \otimes f_b^{(3)} \\ & - \alpha_s^{(4)}\left(Q^2\right) P_{qq}^{(0)} L \otimes f_b^{(4)}\left(Q^2\right) \\ & - \alpha_s^{(4)}\left(Q^2\right) P_{qg}^{(0)} L \otimes f_g^{(4)}\left(Q^2\right) + O\left(\alpha_s^2\right) \\ =& f_h^{(4)}\left(Q^2\right) - \alpha_s^{(4)}\left(Q^2\right)\left[K_{bb}^{(1)}\left(m^2\right) + P_{qq}^{(0)} L\right] \otimes f_h^{(4)}\left(Q^2\right) \\ & - \alpha_s^{(4)}\left(Q^2\right) P_{qg}^{(0)} L \otimes f_g^{(4)}\left(Q^2\right) + O\left(\alpha_s^2\right), \end{aligned} \tag{45}$$

where as usual $L \equiv \frac{Q^2}{m^2}$ and $P_{ij}^{(0)}$ are the leading order splitting functions. From this last equation we notice that, even if $f_h^{(3)}$ is scale independent, writing it in terms of 4FS PDFs and expanding out K_{ij}, since we retain only terms up to $O(\alpha_s)$, we induce a subleading dependence on the scale Q^2.

In order to find a simple expression for the term ΔF up to order $O(\alpha_s)$ we notice that we can relate the 3FS coefficient functions in the massless limit to the 4FS mass-independent ones. In fact, order by order in the same strong coupling, the equation below has to be satisfied by definition

$$C_i^{(3,0)} \otimes f_i^{(3)} = C_i^{(4)} \otimes f_i^{(4)}\left(m^2\right), \tag{46}$$

with $i = h,\bar{h}$. Thus using standard evolution equations to evolve from scale m^2 to scale Q^2 together with eq.(44) we get

$$C_i^{(3,0)} \otimes f_i^{(3)} = C_i^{(4)} \otimes \left(f_i^{(3)} + \alpha_s K_{hh}^{(1)}\left(m^2\right) \otimes f_h^{(3)} + \alpha_s L P_{qq}^{(0)} \otimes f_h^{(4)} + O\left(\alpha_s^2\right)\right), \tag{47}$$

from which we read

$$\begin{aligned} C_i^{(3,0),0} &= C_i^{(4),0} \\ C_i^{(3,0),1} &= C_i^{(4),1} + C_i^{(4),0} \otimes \left(K_{hh}^{(1)}\left(m^2\right) + P_{qq}^{(0)} L\right). \end{aligned} \tag{48}$$

Substituiting eq.(45) and (48) in eq.(40) we find

$$\Delta F_h\left(x,Q^2\right)=\sum_{i=h,\bar h}\left[C_i^{(3),0}\left(\frac{Q^2}{m^2}\right)-C_i^{(4),0}\right]\otimes f_i^{(4)}\left(Q^2\right)$$
$$+\,\alpha_s^{(4)}\left(Q^2\right)\sum_{i=h,\bar h}\left[C_i^{(3),1}\left(\frac{Q^2}{m^2}\right)-C_i^{(4),1}-C_i^{(4),0}\otimes\left(K_{hh}^{(1)}\left(m^2\right)+P_{qq}^{(0)}L\right)\right]\otimes f_i^{(4)}\left(Q^2\right)$$
$$-\,\alpha_s^{(4)}\left(Q^2\right)\sum_{i=h,\bar h}\left[C_i^{(3),0}\left(\frac{Q^2}{m^2}\right)-C_i^{(4),0}\right]\otimes LP_{qg}^{(0)}\otimes f_g^{(4)}\left(Q^2\right)+O\left(\alpha_s^2\right). \tag{49}$$

The final result for the heavy component of the structure function in DIS can be obtained adding up Eq. (39) of the previous section and Eq. (49), obtaining, after a certain amount of cancellations

$$F_h\left(x,Q^2\right)=\sum_{i=h,\bar h}C_i^{(3),0}\left(\frac{Q^2}{m^2}\right)\otimes f_i^{(4)}\left(Q^2\right)$$
$$+\,\alpha_s^{(4)}\left(Q^2\right)\sum_{i=h,\bar h}\left[C_i^{(3),1}\left(\frac{Q^2}{m^2}\right)-C_i^{(3),0}\left(\frac{Q^2}{m^2}\right)\otimes\left(K_{hh}^{(1)}\left(m^2\right)+P_{qq}^{(0)}L\right)\right]\otimes f_i^{(4)}\left(Q^2\right)$$
$$+\,\alpha_s^{(4)}\left(Q^2\right)\left[C_g^{(3),1}\left(\frac{Q^2}{m^2}\right)-\sum_{i=h,\bar h}C_i^{(3),0}\left(\frac{Q^2}{m^2}\right)\otimes P_{qg}^{(0)}L\right]\otimes f_g^{(4)}\left(Q^2\right)+O\left(\alpha_s^2\right). \tag{50}$$

We note that the result reduces to the expression obtained combining massless PDFs $f_i^{(4)}$, evolved in 4FS, with the massive coefficients $C_i^{(3)}$ and subtracting from the latter the unresummed logarithms. This means that, since we are considering PDFs also for the heavy flavour, we are resumming to all order the collinear logarithms coming from heavy quarks emissions, and then we are removing the logarithmic terms contained also in the 3FS expressions, which have already been resummed introducing heavy quark PDFs. This approach, which here is obtained in the context of FONLL scheme, is known as ACOT scheme, Ref. [13], [14].

As previously done in the standard FONLL scheme, we can write the result as a sum of the 3FS and a difference term, obtained by subtracting the massless limit of the 3FS to the 4FS. We have seen in Eq. (35) as this difference term is subleading in the region where L is not large, as regards the standard FONLL.

Re-writing $F_h^{(d)}$ in the generalized FONLL method we find

$$\begin{aligned} F_h^{(d)} &= \sum_{i=h,\bar{h}} \left(C_i^{(4),0} + \alpha_s^{(4)}\left(Q^2\right) C_i^{(4),1} \right) \otimes f_i^{(4)}\left(Q^2\right) \\ &- \alpha_s^{(4)}\left(Q^2\right) \sum_{i=h,\bar{h}} C_i^{(4),0} \otimes LP_{qg}^{(0)}) \otimes f_g^{(4)}\left(Q^2\right) \\ &- \sum_{i=h,\bar{h}} \left(C_i^{(3,0),0} + C_i^{(3,0),1} \alpha_s^{(4)}\left(Q^2\right) \right) \otimes f_i^{(3)}, \end{aligned} \tag{51}$$

where the first two lines of the equation above are the difference term of the standard FONLL, while the last one is the new contribution coming from the presence of an intrinsic charm PDF. While we have already verified that, when using the expansion eq.(36), the first two lines of the expression above are subleading in α_s, it turns out that in the actual case, using only the matching condition given by Eq. (45), the term above vanishes identically without using the expansion Eq. (36). The reason for that is Eq. (45), where re-expressing the 3FS PDFs in terms of the 4FS ones, the difference in evolution is only compensated up to $O\left(\alpha_s\right)$, so that the higher order collinear log which should appear in the equation above, at order α_s are subtracted off, leading to a not only subleading but identically vanishing difference term.

Therefore, since the difference term vanishes, the final result for the generalized FONLL up to $O\left(\alpha\right)$ can be written in the simple form below:

$$\begin{aligned} F_h^{FONLL} &= \sum_{i=q,\bar{q},g} C_i^{(3)}\left(\frac{Q^2}{m^2}\right) \otimes f_i^{(3)}\left(Q^2\right) + \sum_{i=h,\bar{h}} C_i^{(3)}\left(\frac{Q^2}{m^2}\right) \otimes f_i^{(3)} \\ &= \sum_{i,j=q,\bar{q},h,\bar{h},g} C_i^{(3)}\left(\frac{Q^2}{m^2}\right) \otimes K_{ij}^{-1}\left(Q^2\right) \otimes f_j^{(4)}\left(Q^2\right). \end{aligned} \tag{52}$$

Thus we have presented explicit results up to $O\left(\alpha_s\right)$ for the generalization of FONLL scheme in DIS, to account for the presence of a fitted charm PDF. We find that the outcomes are the same we would obtain using the so-called ACOT scheme, in other words the difference term is not only subleading but identically zero.

The results reported so far are only up $O\left(\alpha_s\right)$, but the discussion can be easily generalized to higher orders. In fact the general expression given in Eq. (41) can be written as

$$\begin{aligned} F\left(x,Q^2\right) = &\sum_{i,j=g,q,\bar{q},h,\bar{h}} \left[C_i^{(3)}\left(\frac{Q^2}{m_h^2}\right) - C_i^{(3,0)}\left(\frac{Q^2}{m_h^2}\right) \right] \otimes K_{ij}^{-1}\left(Q^2\right) \otimes f_j^{(4)}\left(Q^2\right) \\ &+ \sum_{i,j=g,q,\bar{q},h,\bar{h}} C_i^{(4)} \otimes f_i^{(4)}\left(Q^2\right), \end{aligned} \tag{53}$$

where we have used the inverse $K_{ij}^{-1}\left(Q^2\right)$ of the matching matrix of Eq. (45) to express the massive PDFs $f_i^{(3)}$ in terms of the massless ones $f_i^{(4)}$ and where it is understood that all the quantities are expanded to the desired order in power series of $\alpha_s^{(4)}$, with the massless PDFs expressed in terms of a set of PDFs at a reference scale and then evolved through perturbative evolution to scale Q^2. We notice that, using the matching condition of Eq. (44) evolved till scale Q^2, the second and third terms of eq.(53) cancel at each order in $\alpha_s^{(4)}$, leading again at eq.(52). This shows how the vanishing of the difference term is not only a feature of the computations to order $O\left(\alpha_s\right)$, but it regards also higher orders. This feature is discussed further in Ref. [2], where also the phenomenological implication of this outcome are addressed.

4 Heavy quarks in hadronic processes

In the previous sections, we have seen how to apply the FONLL method to the calculation of structure functions in deep-inelastic scattering. First we have assumed no intrinsic component for heavy quarks, then we have shown as FONLL scheme can be generalized to account for a fitted heavy quark PDF. The aim of this section is to present the application of FONLL method to a hadronic process, taking as specific example the Higgs production in bottom-quark fusion, focusing on the calculation of the total cross-section. In the first subsection we present the general structure of the FONLL method in a generic hadronic process, while explicit results for the Higgs production in bottom-quark fusion are obtained in the second subsection. The results reported here have been first derived in Ref. [6].

4.1 General structure

In this subsection we describe the general construction of the FONLL method in a generic hadronic process, taking as assumption no intrinsic component of heavy quark. The generalization to the case with a fitted heavy quark PDF will be done in the next section.

The general expression for the hadronic cross section in the massless scheme is

$$\begin{aligned}\sigma^{(n_l+1)} = \int\int dx_1 dx_2 \sum_{ij} f_i^{(n_l+1)}\left(x_1, Q^2\right) f_j^{(n_l+1)}\left(x_2, Q^2\right) \\ \times\, \hat{\sigma}_{ij}^{(n_l+1)}\left(x_1, x_2, \alpha_s^{(n_l+1)}\left(Q^2\right)\right),\end{aligned} \tag{54}$$

where the sum runs over all quarks and antiquarks, both light and heavy. Since this is the massless scheme, the heavy flavour is treated as a massless parton, and it contributes to the running of α_s and to the QCD evolution equations as well.

In the massive scheme the general expression for the hadronic cross section is

$$\begin{aligned}\sigma^{(n_l)} = \int\int dx_1 dx_2 \sum_{ij} f_i^{(n_l)}\left(x_1, Q^2\right) f_j^{(n_l)}\left(x_2, Q^2\right) \\ \times\, \hat{\sigma}_{ij}^{(n_l)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(n_l)}\left(Q^2\right)\right),\end{aligned} \tag{55}$$

where the sum runs only over the light flavours and the gluon. Heavy quarks decouple from the running of coupling constant and from DGLAP equations satisfied by $f_i^{(n_l)}\left(x_2, Q^2\right)$, but, since this is the massive scheme, the full dependence on m is retained in the expression of the partonic cross section $\hat{\sigma}_{ij}^{(n_l)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(n_l)}\left(Q^2\right)\right)$.

To carry on the FONLL method, we follow the same general line presented in the context of DIS. We re-express the massive scheme cross section in terms of the massless PDFs and the coupling $\alpha_s^{(n_l+1)}(Q^2)$, using the matching condition given in Eq. (4) and (3). This gives the expression

$$\begin{aligned}\sigma^{(n_l)} = \int\int dx_1 dx_2 \sum_{ij=g,q,\bar{q}} f_i^{(n_l+1)}\left(x_1, Q^2\right) f_j^{(n_l+1)}\left(x_2, Q^2\right) \\ \times B_{ij}^{(n_l)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(n_l+1)}\left(Q^2\right)\right),\end{aligned} \tag{56}$$

where the coefficient functions B_{ij} are such that replacing in Eq. (56) the matching conditions Eq. (4), (3), one gets the starting expression Eq. (55). The B_{ij} coefficient functions can be expanded as power of $\alpha_s^{(n_l+1)}$ as

$$\begin{aligned}&B_{ij}^{(n_l)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(n_l+1)}\left(Q^2\right)\right) \\ &= \sum_{p=0}^{N} \left(\alpha_s^{(n_l+1)}\left(Q^2\right)\right)^p B_{ij}^{(p)}\left(x_1, x_2, \frac{Q^2}{m^2}\right),\end{aligned} \tag{57}$$

where N is the order of expansion needed to reach the desired accuracy.

As already pointed out, using the DGLAP evolution equations the heavy quark PDFs $f_h^{n_l+1}$ and $f_{\bar{h}}^{n_l+1}$ can be determined in terms of the gluon and the light quark parton distributions convoluted with coefficient function expressed as power series in $\alpha_s^{(n_l+1)}$. This allows us to re-write the massless scheme cross section as

$$\begin{aligned}\sigma^{(n_l+1)} = \int\int dx_1 dx_2 \sum_{ij=g,q,\bar{q}} f_i^{(n_l+1)}\left(x_1, Q^2\right) f_j^{(n_l+1)}\left(x_2, Q^2\right) \\ \times A_{ij}^{(n_l+1)}\left(x_1, x_2, L, \alpha_s^{(n_l+1)}\left(Q^2\right)\right),\end{aligned} \tag{58}$$

with, as before, $L = \log \frac{Q^2}{m^2}$ and where the perturbative expansion of A_{ij} has the general form

$$\begin{aligned}&A_{ij}^{(n_l+1)}\left(x_1, x_2, L, \alpha_s^{(n_l+1)}\left(Q^2\right)\right) \\ &= \sum_{p=0}^{N} \left(\alpha_s^{(n_l+1)}\left(Q^2\right)\right)^p \sum_{k=0}^{\infty} A_{ij}^{(p),(k)}\left(x_1, x_2\right) \left(\alpha_s^{(n_l+1)}\left(Q^2\right) L\right)^k.\end{aligned} \tag{59}$$

The terms contained in both massive and massless scheme are given by

$$\begin{aligned}&B_{ij}^{(0)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(n_l+1)}\right) \\ &= \sum_{p=0}^{N} \left(\alpha_s^{(n_l+1)}\left(Q^2\right)\right)^p B_{ij}^{(0),(p)}\left(x_1, x_2, \frac{Q^2}{m^2}\right),\end{aligned} \tag{60}$$

with $B_{ij}^{(0),(p)}$ providing the massless limit of $B_{ij}^{(p)}$ in the sense that

$$\lim_{m\to 0}\left[B_{ij}^{(p)}\left(x_1,x_2,\frac{Q^2}{m^2}\right)-B_{ij}^{(0),(p)}\left(x_1,x_2,\frac{Q^2}{m^2}\right)\right]=0. \tag{61}$$

According to eq.(11), the double counted terms can be written also as

$$B_{ij}^{(0),(p)}\left(x_1,x_2,\frac{Q^2}{m^2}\right)=\sum_{k=0}^{p}A_{ij}^{(p-k),(k)}\left(x_1,x_2\right)L^k, \tag{62}$$

namely starting from the massless scheme and expanding the heavy quark PDFs in power of $\alpha_s^{(n_l+1)}$, using DGLAP equations. Finally we define the massless limit of the massive scheme cross section as

$$\begin{aligned}\sigma^{(n_l),(0)}=&\int\int dx_1dx_2\sum_{ij=g,q,\bar{q}}f_i^{(n_l)}\left(x_1,Q^2\right)f_j^{(n_l)}\left(x_2,Q^2\right)\\ &\times B_{ij}^{(0)}\left(x_1,x_2,\frac{Q^2}{m^2},\alpha_s^{(n_l)}\left(Q^2\right)\right).\end{aligned} \tag{63}$$

Thus the FONLL method can be expressed as follows: we replace in the massless scheme expression eq.(58) all the massless terms $B_{ij}^{(0),p}$ which appear both in the power expansion of A_{ij}, Eq. (59), and in the expansion of $B_{ij}^{(0)}$, Eq. (60), with their full massive expression $B_{ij}^{(p)}$, Eq. (57). In this way the final result will retain at the massless level the same logarithmic accuracy of the starting massless scheme, and at the massive level the fixed order accuracy in $\alpha_s^{(n_l+1)}$ corresponding to the number of massive orders which have been included in the computation of B_{ij}. Therefore we define

$$\sigma^{FONLL}=\sigma^{(n_l)}+\sigma^{(n_l+1)}-\sigma^{(n_l),(0)}. \tag{64}$$

The procedure presented in this subsection may be applied to any hadronic process, while the specific hard cross sections which appear the expression above depend on the specific process we are considering. Therefore in the following part of this section we will report explicit results for a specific hadronic process, specifying our calculation to it.

4.2 FONLL applied at Higgs production in bottom-quark fusion

In this subsection we will explicitly apply the FONLL method to the specific hadronic process where Higgs boson is produced from bottom-quark fusion. Therefore, we will refer to bottom quark as the massive parton, and we will talk about four and five flavour schemes (4FS and 5FS respectively). After a brief introduction about this particular hadronic process, we will proceed with the explicit implementation of the FONLL method introduced in the previous section.

Higgs boson production in bottom quark fusion, is an hadronic process which has been extensively studied in the past. By the standard model, the Higgs boson has a Yukawa coupling to fermions, which is proportional to m/v, where m is the mass of the fermion coupled to the Higgs boson and $v \sim 246\ GeV$ is the vacuum-expectation value of the Higgs field. Considering the bottom quark, the Yukawa coupling is m_b/v, therefore the cross section for the production of Higgs boson in association with bottom quarks is relatively small compared to other processes for the production of Higgs boson. Despite this, one of the reasons why this process has been studied is that there are models in which the bottom quark Yukawa coupling is enhanced. Such a enhancement occurs, for example, in a two Higgs doublet model, described in Ref. [10]. Here we will take this hadronic process as a specific example of application of the FONLL method, without discussing any of its specific phenomenological features.

We now turn to the implementation of the FONLL method to Higgs production from bottom quark fusion. First we need to relabel the perturbative orders at which FONLL results can be computed. Concerning the application of the FONLL method to DIS, we have previously seen three different possibilities. The simplest one, called FONLL-A, is obtained by computing both massive and massless results at the same lowest possible order in α_s, which, in the case of DIS, is of course $O(\alpha_s)$. In fact the mismatch between LO diagrams of the massive and massless scheme is of one order, as we see from the Feynman diagrams in Fig. (1). With FONLL-B we have defined the situation in which the massive scheme is computed with a precision one order higher than the massless one, so we have a massive calculation up to $O(\alpha_s^2)$ versus a massless accuracy of only α_s. Finally by FONLL-C we have addressed the case in which both massive and massless scheme are computed with the same accuracy, one order higher than the one of FONLL-A. Namely, in the case of DIS, FONLL-A corresponds to NLL-LO, FONLL-B to NLL-NLO, FONLL-C to NNLL-NLO, where by "leading" we always mean the first order at which the results does not vanish.

In the case of Higgs production in bottom-quark fusion, looking at diagrams in Fig. (3) and (4), we see that the mismatch between leading orders in 4FS and 5FS is now by two orders: the leading order in 4FS is $O(\alpha_s^2)$ while the one of the 5FS is $O(\alpha_s^0)$. Therefore, the simplest non trivial case corresponding to the previous defined FONLL-A is now NNLL-LO. In the same way we define FONLL-B as NNLL-NLO and FONLL-C as NNNLL-NLO. Currently, in the 5FS results are known up to NNLO, therefore we could get a NNLL accuracy by combining with NNLO PDFs, while in the 4FS they are known up to NLO. Therefore the current knowledges allow computations of FONLL-A and FONLL-B.

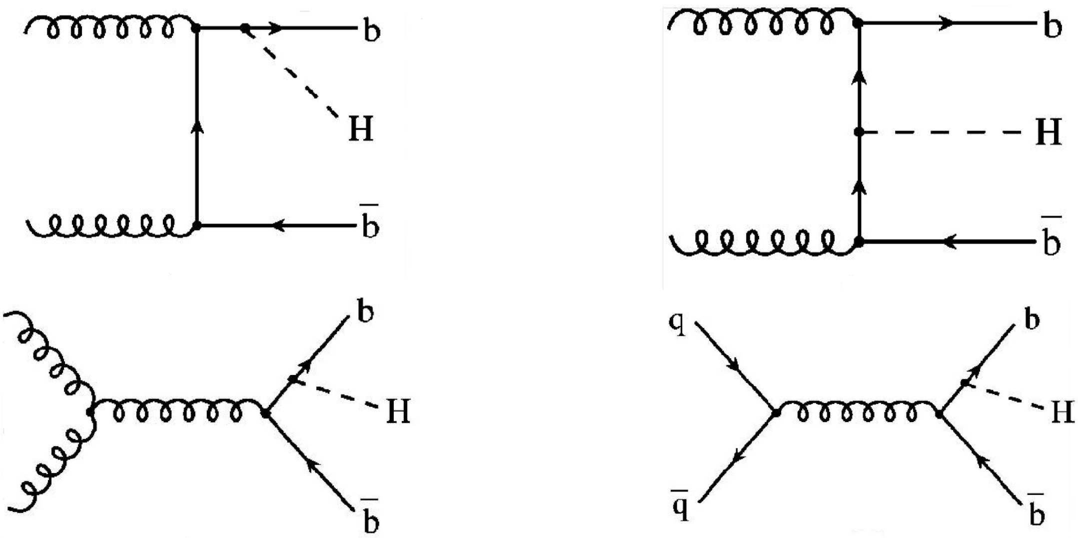

Figure 3: Leading-order contribution to the four flavour scheme.

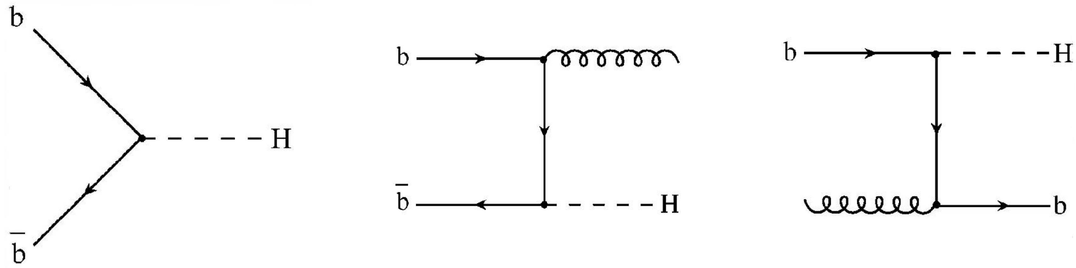

Figure 4: From left to right, leading-order and next-to-leading contributions to the five flavour scheme hard cross sections.

In the following we will work out explicit expressions for Eq. (64) for Higgs production in bottom-quark fusion, in the simplest case FONLL-A. In the 5FS, in order to get a NNLL accuracy, the partonic cross sections have to be computed up to $O(\alpha_s^2)$, therefore we have to consider the following subprocesses:

- $O(\alpha_s^0) \Rightarrow b\bar{b} \to h$;
- $O(\alpha_s^1) \Rightarrow b\bar{b} \to h$ with 1-loop corrections, real gluon emissions $b\bar{b} \to hg$ and processes with a gluon in the initial state $bg \to bh$ at tree level;
- $O(\alpha_s^2) \Rightarrow b\bar{b} \to h$ with 2-loop corrections, real gluon emissions $b\bar{b} \to hg$ and processes with a gluon in the initial state $bg \to bh$ with 1-loop corrections, tree level processes $bq \to hbq$, $gg \to hb\bar{b}$, $bb \to hb\bar{b}$ and $qq \to hb\bar{b}$.

Thus the relevant perturbative orders in each parton channel are given by

$$
\begin{aligned}
&\hat{\sigma}^{(5)}_{b\bar{b}}\left(x_1,x_2,\alpha_s^{(5)}\left(Q^2\right)\right)\\
&=\hat{\sigma}^{(5),0}_{b\bar{b}}\left(x_1,x_2\right)+\alpha_s^{(5)}\left(Q^2\right)\hat{\sigma}^{(5),1}_{b\bar{b}}\left(x_1,x_2\right)+\left(\alpha_s^{(5)}\left(Q^2\right)\right)^2\hat{\sigma}^{(5),2}_{b\bar{b}}\left(x_1,x_2\right)+O\left(\alpha_s^3\right)\\
&\hat{\sigma}^{(5)}_{bg}\left(x_1,x_2,\alpha_s^{(5)}\left(Q^2\right)\right)\\
&=\alpha_s^{(5)}\left(Q^2\right)\hat{\sigma}^{(5),1}_{bg}\left(x_1,x_2\right)+\left(\alpha_s^{(5)}\left(Q^2\right)\right)^2\hat{\sigma}^{(5),2}_{bg}\left(x_1,x_2\right)+O\left(\alpha_s^3\right)\\
&\hat{\sigma}^{(5)}_{bq}\left(x_1,x_2,\alpha_s^{(5)}\left(Q^2\right)\right)=\left(\alpha_s^{(5)}\left(Q^2\right)\right)^2\hat{\sigma}^{(5),2}_{bq}\left(x_1,x_2\right)+O\left(\alpha_s^3\right)\\
&\hat{\sigma}^{(5)}_{gg}\left(x_1,x_2,\alpha_s^{(5)}\left(Q^2\right)\right)=\left(\alpha_s^{(5)}\left(Q^2\right)\right)^2\hat{\sigma}^{(5),2}_{gg}\left(x_1,x_2\right)+O\left(\alpha_s^3\right)\\
&\hat{\sigma}^{(5)}_{bb}\left(x_1,x_2,\alpha_s^{(5)}\left(Q^2\right)\right)=\left(\alpha_s^{(5)}\left(Q^2\right)\right)^2\hat{\sigma}^{(5),2}_{bb}\left(x_1,x_2\right)+O\left(\alpha_s^3\right)\\
&\hat{\sigma}^{(5)}_{q\bar{q}}\left(x_1,x_2,\alpha_s^{(5)}\left(Q^2\right)\right)=\left(\alpha_s^{(5)}\left(Q^2\right)\right)^2\hat{\sigma}^{(5),2}_{q\bar{q}}\left(x_1,x_2\right)+O\left(\alpha_s^3\right).
\end{aligned}
\tag{65}
$$

Concerning 4FS, in FONLL-A scheme results are included up to the first non trivial order, namely $O\left(\alpha_s^2\right)$. Therefore in 4FS expression we can simply replace the 4FS parameters $\alpha_s^{(4)}$ and $f_i^{(4)}$ with the corresponding 5FS quantities $\alpha_s^{(5)}$ and $f_i^{(5)}$ as their difference is higher order in α_s. Therefore we have

$$
B_{ij}\left(x_1,x_2,\frac{Q^2}{m^2},\alpha_s\left(Q^2\right)\right)=\hat{\sigma}^{(4)}_{ij}\left(x_1,x_2,\frac{Q^2}{m^2},\alpha_s\left(Q^2\right)\right)+O\left(\alpha_s^3\right). \tag{66}
$$

We now work out the massless limit of the massive scheme. Since this limit starts at order $O\left(\alpha_s^2\right)$, using the general expression given by Eq. (62) we have

$$
\begin{aligned}
&B^{(0)}_{ij}\left(x_1,x_2,L,\alpha_s\left(Q^2\right)\right)\\
&=\left(\alpha_s\right)^2B^{(0),(2)}_{ij}\left(x_1,x_2,L\right)+O\left(\alpha_s^3\right)\\
&=\left(\alpha_s\right)^2\left(A^{(2),(0)}_{ij}\left(x_1,x_2\right)+A^{(1),(1)}_{ij}\left(x_1,x_2\right)L+A^{(0),(2)}_{ij}\left(x_1,x_2\right)L^2\right)+O\left(\alpha_s^3\right)
\end{aligned}
\tag{67}
$$

We find the general form of $A^{(p),(k)}_{ij}$ coefficients starting from the 5FS expression and expanding the bottom quark PDF in powers of α_s, as already discussed. Recalling Eq. (36)

$$
f_b\left(x,Q^2\right)=\alpha_s\left(Q^2\right)L\int_x^1\frac{dy}{y}P_{qg}\left(y\right)f_g\left(\frac{x}{y},Q^2\right)+O\left(\alpha_s^2\right), \tag{68}
$$

where

$$P_{qg}(y) = \frac{T_R}{2\pi}\left[y^2 + (1-y)^2\right], \tag{69}$$

and keeping in mind the relevant perturbative orders of Eq. (65) we can write

$$\sum_{i,j=q,\bar{q},b,\bar{b},g} \int_0^1 \int_0^1 dx_1\, dx_2\, f_i^{(5)}\left(x_1, Q^2\right) f_j^{(5)}\left(x_2, Q^2\right) \hat{\sigma}_{ij}^{(5)}\left(x_1, x_2, \alpha_s^{(5)}\left(Q^2\right)\right)$$

$$\begin{aligned}
&= \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 \int_0^1 \int_0^1 dx_1\, dx_2 \left[f_q^{(5)}\left(x_1, Q^2\right) f_{\bar{q}}^{(5)}\left(x_2, Q^2\right) + (x_1 \leftrightarrow x_2)\right] \hat{\sigma}_{q\bar{q}}^{(5),2}(x_1, x_2) \\
&+ \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 \int_0^1 \int_0^1 dx_1\, dx_2 \left[f_g^{(5)}\left(x_1, Q^2\right) f_g^{(5)}\left(x_2, Q^2\right) + (x_1 \leftrightarrow x_2)\right] \hat{\sigma}_{gg}^{(5),2}(x_1, x_2) \\
&+ \left(\alpha_s^{(5)}\left(Q^2\right)\right) \int_0^1 \int_0^1 dx_1\, dx_2 \left[f_g^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) + (x_1 \leftrightarrow x_2)\right] \hat{\sigma}_{bg}^{(5),1}(x_1, x_2) \\
&+ \left(b \to \bar{b}\right) \\
&+ \int_0^1 \int_0^1 dx_1\, dx_2 \left[f_b^{(5)}\left(x_1, Q^2\right) f_{\bar{b}}^{(5)}\left(x_2, Q^2\right) + (x_1 \leftrightarrow x_2)\right] \hat{\sigma}_{b\bar{b}}^{(5),0}(x_1, x_2),
\end{aligned} \tag{70}$$

from which we get, using Eq. (36)

$$\begin{aligned}
&\left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 \int_0^1 \int_0^1 dx_1\, dx_2 \left[f_q^{(5)}\left(x_1, Q^2\right) f_{\bar{q}}^{(5)}\left(x_2, Q^2\right) + (x_1 \leftrightarrow x_2)\right] \hat{\sigma}_{q\bar{q}}^{(5),2}(x_1, x_2) \\
&+ \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 \int_0^1 \int_0^1 dx_1\, dx_2 \left[f_g^{(5)}\left(x_1, Q^2\right) f_g^{(5)}\left(x_2, Q^2\right) + (x_1 \leftrightarrow x_2)\right] \hat{\sigma}_{gg}^{(5),2}(x_1, x_2) \\
&+ \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 L \int_0^1 \int_0^1 dx_1\, dx_2 \left[f_g\left(x_1, Q^2\right) \int_{x_2}^1 \frac{dy}{y} P_{qg}(y) f_g\left(\frac{x_2}{y}, Q^2\right) + (x_1 \leftrightarrow x_2)\right] \\
&\qquad \times \hat{\sigma}_{bg}^{(5),1}(x_1, x_2) + \left(b \to \bar{b}\right) \\
&+ \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 L^2 \int_0^1 \int_0^1 dx_1\, dx_2 [\int_{x_1}^1 \frac{dy_1}{y_1} P_{qg}(y_1) f_g\left(\frac{x_1}{y_1}, Q^2\right) \\
&\qquad \times \int_{x_2}^1 \frac{dy_2}{y_2} P_{qg}(y_2) f_g\left(\frac{x_2}{y_2}, Q^2\right) + (x_1 \leftrightarrow x_2)] \hat{\sigma}_{b\bar{b}}^{(5),0}(x_1, x_2).
\end{aligned} \tag{71}$$

The third and fourth terms of the previous equation can be rewritten as

$$\begin{aligned}
&\int_0^1 \int_0^1 dx_1\, dx_2\, f_g\left(x_1, Q^2\right) f_g\left(x_2, Q^2\right) \\
&\times \int_0^1 dy P_{qg}(y) \left[\hat{\sigma}_{bg}^{(5),1}(x_1, y x_2) +) \hat{\sigma}_{bg}^{(5),1}(y x_1, x_2)\right] + \left(b \to \bar{b}\right),
\end{aligned} \tag{72}$$

and

$$\int_0^1 \int_0^1 dx_1\, dx_2 \left[f_g\left(x_1, Q^2\right) f_g\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right]$$
$$\int_0^1 \int_0^1 dy_1 dy_2 P_{qg}\left(y_1\right) P_{qg}\left(y_2\right) \times \hat{\sigma}_{b\bar{b}}^{(5),0}\left(y_1 x_1, y_2 x_2\right)$$
$$= 2 \int_0^1 \int_0^1 dx_1\, dx_2 \left[f_g\left(x_1, Q^2\right) f_g\left(x_2, Q^2\right)\right] \int_0^1 \int_0^1 dy_1 dy_2 P_{qg}\left(y_1\right) P_{qg}\left(y_2\right)$$
$$\times \hat{\sigma}_{b\bar{b}}^{(5),0}\left(y_1 x_1, y_2 x_2\right), \tag{73}$$

therefore from the relations above we read the non vanishing terms $A_{ij}^{(p),(k)}$:

$$A_{qq}^{(2),(0)}\left(x_1, x_2\right) = \hat{\sigma}_{q\bar{q}}^{(5),2}\left(x_1, x_2\right) + \left(x_1 \leftrightarrow x_2\right) \tag{74}$$

$$A_{gg}^{(2),(0)}\left(x_1, x_2\right) = \hat{\sigma}_{gg}^{(5),2}\left(x_1, x_2\right) + \left(x_1 \leftrightarrow x_2\right) \tag{75}$$

$$A_{gg}^{(1),(1)}\left(x_1, x_2\right) = \int_0^1 dy P_{qg}\left(y\right) \left[\hat{\sigma}_{bg}^{(5),1}\left(x_1, y x_2\right) + \hat{\sigma}_{bg}^{(5),1}\left(y x_1, x_2\right)\right] + \left(b \rightarrow \bar{b}\right) \tag{76}$$

$$A_{gg}^{(0),(2)}\left(x_1, x_2\right) = 2 \int_0^1 \int_0^1 dy_1 dy_2 P_{qg}\left(y_1\right) P_{qg}\left(y_2\right) \hat{\sigma}_{b\bar{b}}^{(5),0}\left(y_1 x_1, y_2 x_2\right). \tag{77}$$

In order to write down the explicit FONLL-A expression we have introduced the formal expansion

$$\sigma^{FONLL-A} = \sigma^{FONLL-A,(0)} + \alpha_s\left(Q^2\right) \sigma^{FONLL-A,(1)} + \left(\alpha_s\left(Q^2\right)\right)^2 \sigma^{FONLL-A,(2)} + O\left(\alpha_s^3\right). \tag{78}$$

In doing this expansion only the coefficients $B_{ij}^{(4)}$, $B_{ij}^{(0)}$ and $A_{ij}^{(5)}$ of Eqs. (57), (60) and (59) respectively are expanded, but not the heavy quark PDFs. Namely the perturbative orders of Eq.(78) concerns only the partonic cross sections, not the heavy quark PDFs. Therefore, the contribution of order $O\left(\alpha_s^0\right)$ in Eq. (78) really starts at $O\left(\alpha_s^2\right)$ once one uses the heavy quark PDFs expansion of Eq. (68), as it has to be done for matching the 4FS. Using Eq. (64) we have

$$\sigma^{FONLL-A,(0)} = \int\int dx_1 dx_2 \left[f_b^{(5)}\left(x_1, Q^2\right) f_{\bar{b}}^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{b\bar{b}}^{(5),(0)}\left(x_1, x_2\right); \tag{79}$$

$$\sigma^{FONLL-A,(1)} =$$
$$\int\int dx_1 dx_2 \left[f_b^{(5)}\left(x_1, Q^2\right) f_{\bar{b}}^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{b\bar{b}}^{(5),(1)}\left(x_1, x_2\right)$$
$$+ \int\int dx_1 dx_2 \left[f_g^{(5)}\left(x_1, Q^2\right) f_{\bar{b}}^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right) + \left(b \rightarrow \bar{b}\right)\right] \hat{\sigma}_{gb}^{(5),(1)}\left(x_1, x_2\right). \tag{80}$$

The $O\left(\alpha_s^0\right)$ and $O\left(\alpha_s\right)$ contributions of Eq. (79) and (80) coincide with the massless expressions, since as for partonic cross sections the massive expressions start being non-zero only at $O\left(\alpha_s^2\right)$. As for the $O\left(\alpha_s^2\right)$ contribution it can be written as

$$\sigma^{FONLL-A,(2)} = \sigma^{(4),(2)} + \sigma^{(d),(2)}, \tag{81}$$

where

$$\sigma^{(d),(2)} = \sigma^{(5),(2)} - \sigma^{(4),(0),(2)}, \tag{82}$$

and $\sigma^{(4),(0),(2)}$ is written in terms of the quantities previously computed

$$\sigma^{(4),(0),(2)} = \int\int dx_1 dx_2 \sum_{ij=q,g} f_i^{(5)}\left(x_1, Q^2\right) f_j^{(5)}\left(x_2, Q^2\right) B_{ij}^{(0),(2)}\left(x_1, x_2, L, \alpha_s\right). \tag{83}$$

Therefore we get

$$\begin{aligned}
&\sigma^{(d),(2)} = \int\int dx_1 dx_2 \\
&\left[f_b^{(5)}\left(x_1, Q^2\right) f_{\bar{b}}^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{b\bar{b}}^{(5),2} \\
&+ f_b^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) \hat{\sigma}_{bb}^{(5),2} \\
&+ \left[f_g^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{gb}^{(5),2} + \left(b \to \bar{b}\right) \\
&+ \left[f_q^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{qb}^{(5),2} + \left(b \to \bar{b}, q \to \bar{q}\right) \\
&- L \int_0^1 dy P_{qg}\left(y\right) \left[\hat{\sigma}_{bg}^{(5),1}\left(x_1, y x_2\right) + \hat{\sigma}_{bg}^{(5),1}\left(y x_1, x_2\right) + \left(b \to \bar{b}\right)\right] f_g^{(5)}\left(x_1, Q^2\right) f_g^{(5)}\left(x_2, Q^2\right) \\
&- 2L^2 \int_0^1 \int_0^1 dy_1 dy_2 P_{qg}\left(y_1\right) P_{qg}\left(y_2\right) f_g^{(5)}\left(x_1, Q^2\right) f_g^{(5)}\left(x_2, Q^2\right) \hat{\sigma}_{b\bar{b}}^{(5),0}\left(y_1 x_1, y_2 x_2\right).
\end{aligned} \tag{84}$$

Finally, in conclusion of this section, we observe that, as expected, the differnce term, whose explicit expression is given by

$$\sigma^{(d)} = \sigma^{(5),(0)} + \alpha_s\left(Q^2\right) \sigma^{(5),(1)} + \left(\alpha_s\left(Q^2\right)\right)^2 \sigma^{(d),(2)}, \tag{85}$$

using the expansion of the heavy quark PDFs, Eq. (68), is $O\left(\alpha_s^3\right)$, namely subleading with respect to the LO computation in the 4FS.

To sum up, in this section we have seen how to apply the FONLL method first to a generic hadronic process, then we have specialized the computation in order to address Higgs production in bottom quark fusion. After relabelling the previously defined perturbative orders at which FONLL could be implemented, we have worked out the specific results of Ref. [6], observing that, as for the terms of order $O\left(\alpha_s^0\right)$ and $O\left(\alpha_s^1\right)$ they coincide with the 5FS results. Therefore, the difference term obtained from the difference between the 5FS and the massless limit of the 4FS, starts being different from the 5FS result only at $O\left(\alpha_s^2\right)$, and, as for the case of DIS, when $Q^2 \sim m_b^2$ it is subleading. In the next sections of this work we will show how to generalize these results in order to account the presence of initial state heavy quarks also in the massive scheme.

5 Initial-state heavy quarks in hadronic processes

In the previous section we have shown how to consistently match the four- and five- flavor scheme computations regarding the total cross section for a generic hadronic process. We assumed the absence of an intrinsic heavy quark component, namely we assumed that in the 4FS, heavy quarks do not appear in the initial states. We have then specialized the results to the case of Higgs production in bottom-quark fusion. The aim of this section is to generalize these results accounting for the presence of a heavy quark in the initial state also in the massive scheme. First we address the general problem, providing equations written in term of partonic cross sections. Then, we explicitly work out analytical expression for them, considering the specific case of Higgs production in bottom-quark fusion.

5.1 General problem

In this subsection we first recall some of the motivations of this work, and then we depict the general lines of the problem, working out the analytical expressions for our results.

As already discussed studying the analogue generalization in the case of deep inelastic scattering, there are several motivations for this. Firstly an intrinsic heavy quark component may well be non-zero, and this is particularly interesting in the case of processes involving charm quark. However, even if the intrinsic heavy quark is definitely zero, having a formalism which allows us to take into account an heavy quark in the initial state is still useful. In fact in the massless scheme the heavy quark is assigned a parton distribution, which is generated perturbatively starting from some vanishing boundary condition. Therefore the general result will depend on the arbitrary choice of the scale at which this perturbative boundary is imposed. Introducing a formalism which takes into account a fitted heavy quark in the initial state would allow us to reabsorb, in the initial condition, the dependence on the arbitrary starting scale of the perturbative component. In addition to this, with a fitted heavy quark initial state we wouldn't have the uncertainty coming from finite order matching condition, which may be relevant in computation at low orders. These are the main reason for the following generalization, since in the example we are going to address, Higgs production in bottom quark fusion, an intrinsic bottom component is likely to be quite small.

We proceed to generalize the FONLL scheme, following the guidelines introduced in the case of DIS. As for 5FS, we have to change only the boundary conditions: while in the standard FONLL the massless PDFs were determined starting from a vanishing boundary condition, now we take as initial condition a fitted PDF at a scale m^2. Then, for $Q^2 > m^2$ the PDFs satisfy DGLAP equations with 5 active flavours. Therefore, in the 5FS the cross section is given by

$$\sigma^{(5)} = \int dx_1 dx_2 \sum_{i,j} f_i^{(5)}\left(x_1, Q^2\right) f_j^{(5)}\left(x_2, Q^2\right) \hat{\sigma}_{ij}^{(5)}\left(x_1, x_2, \alpha_s^{(5)}\left(Q^2\right)\right), \tag{86}$$

with $i, j = q, \bar{q}, b, \bar{b}, g$ and $f_b^{(5)}\left(x_1, Q^2\right)$, $f_{\bar{b}}^{(5)}\left(x_2, Q^2\right)$ worked out by evolution equations with 5 active flavour starting from an appropriate fit performed at a some initial scale m^2. As already discussed, it does not really matter if these PDFs are of perturbative or intrinsic origin. In the former case they will be obtained evolving first the gluon and light quark PDFs fitted at some initial scale u to a chosen energy in 4FS, and then up to Q^2 in the 5FS; in the latter case we will also have an explicit parametrization for an intrinsic component. In the example of the Higgs production in bottom quark fusion, they are likely to be of perturbative origin.

Concerning 4FS, it does change, since we have to introduce heavy quark PDFs at all scale: therefore, for scales $Q^2 < m^2$, we define $f_b^{(4)}\left(m^2\right)$ and $f_{\bar{b}}^{(4)}\left(m^2\right)$ as the bottom quark PDFs in the 4FS. Since in this range of energies the heavy quark is treated as a massive object, it decouples from DGLAP equations to which only four flavours contribute, so this PDF is scale independent. Therefore in the 4FS new contributions to the cross section arise, due to the presence of heavy quarks also in the initial state. In the massive scheme the cross section is

$$\begin{aligned}
\sigma^{(4)} &= \int dx_1 dx_2 \sum_{i,j} f_i^{(4)}\left(x_1, Q^2\right) f_j^{(4)}\left(x_2, Q^2\right) \hat{\sigma}_{ij}^{(4)}\left(x_1, x_2, \frac{Q^2}{m_b^2}, \alpha_s^{(4)}\left(Q^2\right)\right) \\
&= \int dx_1 dx_2 \sum_{i,j=q,\bar{q},g} f_i^{(4)}\left(x_1, Q^2\right) f_j^{(4)}\left(x_2, Q^2\right) \hat{\sigma}_{ij}^{(4)}\left(x_1, x_2, \frac{Q^2}{m_b^2}, \alpha_s^{(4)}\left(Q^2\right)\right) \\
&+ \int dx_1 dx_2 \sum_{k=b,\bar{b}} \sum_{i=q,\bar{q},g} \Big[f_k^{(4)}\left(x_1\right) f_i^{(4)}\left(x_2, Q^2\right) \\
&\qquad + \left(x_1 \to x_2\right)\Big] \hat{\sigma}_{ki}^{(4)}\left(x_1, x_2, \frac{Q^2}{m_b^2}, \alpha_s^{(4)}\left(Q^2\right)\right) \\
&+ \int dx_1 dx_2 \sum_{h,k=b,\bar{b}} f_k^{(4)}\left(x_1\right) f_h^{(4)}\left(x_2\right) \hat{\sigma}_{hk}^{(4)}\left(x_1, x_2, \frac{Q^2}{m_b^2}, \alpha_s^{(4)}\left(Q^2\right)\right),
\end{aligned} \tag{87}$$

where the 4FS PDFs for b and $\bar{b}$ quarks do not depend on the scale Q^2. We define a correction term $\Delta\sigma$ which has to be added to standard FONLL in order to obtained the desired generalization. Therefore the generalized FONLL prescription reads

$$\sigma^{FONLL} = \sigma^{FNRL} + \sum_{i,j=b,\bar{b}} \Delta\sigma_{(ij),1} + \sum_{k=b,\bar{b}} \sum_{i=q,\bar{q},g} \Delta\sigma_{(ki),2} \tag{88}$$

where σ^{FNRL} is the result of the previous section and

$$\begin{aligned}
\Delta\sigma_{(ij),1} &= \int dx_1 dx_2 \, f_i^{(4)}(x_1) \, f_j^{(4)}(x_2) \left[\hat{\sigma}_{ij}^{(4)} - \hat{\sigma}_{ij}^{(4,0)}\right] \\
\Delta\sigma_{(ki),2} &= \int \, dx_1 dx_2 \, \left[f_k^{(4)}(x_1) \, f_i^{(4)}\left(x_2, Q^2\right) + (x_1 \leftrightarrow x_2)\right] \left[\hat{\sigma}_{ki}^{(4)} - \hat{\sigma}_{ki}^{(4,0)}\right].
\end{aligned} \tag{89}$$

Since the 5FS remains unaffected by the introduction of initial-state heavy quarks in the 4FS, only the massive partonic cross sections $\hat{\sigma}^{(4)}$ and their massless limits $\hat{\sigma}^{(4,0)}$ appear in the correction terms $\Delta\sigma$. Eq. (89) is for hadronic processes the analogue of Eq. (40) for DIS.

In this subsection we have presented the general structure through which the FONLL scheme could be modified in treating an hadronic process with initial state heavy quark in the massive scheme. The above correction terms have then to be computed at the desired perturbative order.

5.2 Order α_s

After giving the general structure of the method, we now work out explicitly its analytical expression up $O(\alpha_s)$. This accuracy refers to the perturbative order to which the partonic cross section are computed, without expanding also the PDFs, as done writing eq.(78) in the previous section. We will first identify the relevant subprocesses we have to consider up to this order, then, in order to express correction terms $\Delta\sigma$ in term of 5FS quantities, we will work out the matching condition needed in this case. Finally, as done in eq.(48) for DIS, we will find general relations between the massless scheme partonic cross sections and the massless limit of the massive ones, which allow us to find some general results, presented at the end of this section.

Up to order $O(\alpha_s)$ the relevant subprocesses are

- $O(\alpha_s^0) \Rightarrow b\bar{b} \to h$
- $O(\alpha_s^1) \Rightarrow b\bar{b} \to h$ with 1-loop corrections, real gluon emissions $b\bar{b} \to hg$ and processes with a gluon in the initial state $bg \to bh$, $\bar{b}g \to \bar{b}h$ at tree level

whose relevant perturbative orders are given by

$$\begin{aligned}\hat{\sigma}_{b\bar{b}}^{(4)}\left(x_1,x_2,\frac{Q^2}{m_b^2},\alpha_s^{(4)}\left(Q^2\right)\right) &= \hat{\sigma}_{b\bar{b}}^{(4),0}\left(x_1,x_2,\frac{Q^2}{m_b^2},\left(Q^2\right)\right)\\ &+\hat{\sigma}_{b\bar{b}}^{(4),1}\left(x_1,x_2,\frac{Q^2}{m_b^2},\left(Q^2\right)\right)\alpha_s^{(4)}\left(Q^2\right)+\\ &+\hat{\sigma}_{b\bar{b}}^{(4),2}\left(x_1,x_2,\frac{Q^2}{m_b^2},\left(Q^2\right)\right)\left(\alpha_s^{(4)}\left(Q^2\right)\right)^2+O\left(\alpha_s^3\right),\end{aligned} \tag{90}$$

and

$$\begin{aligned}\hat{\sigma}_{bg}^{(4)}\left(x_1,x_2,\frac{Q^2}{m_b^2},\alpha_s^{(4)}\left(Q^2\right)\right) &= \hat{\sigma}_{bg}^{(4),1}\left(x_1,x_2,\frac{Q^2}{m_b^2},\left(Q^2\right)\right)\alpha_s^{(4)}\left(Q^2\right)+\\ &+\hat{\sigma}_{bg}^{(4),2}\left(x_1,x_2,\frac{Q^2}{m_b^2},\left(Q^2\right)\right)\left(\alpha_s^{(4)}\left(Q^2\right)\right)^2+O\left(\alpha_s^3\right)\\ \hat{\sigma}_{\bar{b}g}^{(4)}\left(x_1,x_2,\frac{Q^2}{m_b^2},\alpha_s^{(4)}\left(Q^2\right)\right) &= \hat{\sigma}_{\bar{b}g}^{(4),1}\left(x_1,x_2,\frac{Q^2}{m_b^2},\left(Q^2\right)\right)\alpha_s^{(4)}\left(Q^2\right)+\\ &+\hat{\sigma}_{\bar{b}g}^{(4),2}\left(x_1,x_2,\frac{Q^2}{m_b^2},\left(Q^2\right)\right)\left(\alpha_s^{(4)}\left(Q^2\right)\right)^2+O\left(\alpha_s^3\right).\end{aligned} \tag{91}$$

Thus the new contributions at the massive scheme up to order $O\left(\alpha_s\right)$ are given by

$$\begin{aligned}&\int dx_1dx_2\,\left[f_b^{(4)}\left(x_1\right)f_{\bar{b}}^{(4)}\left(x_2\right)+\left(x_1\to x_2\right)\right]\hat{\sigma}_{b\bar{b}}^{(4)}\left(x_1,x_2,\frac{Q^2}{m_b^2},\alpha_s^{(4)}\left(Q^2\right)\right)\\ +&\int dx_1dx_2\,\left[f_b^{(4)}\left(x_1\right)f_g^{(4)}\left(x_2,Q^2\right)+\left(x_1\to x_2\right)\right]\hat{\sigma}_{bg}^{(4)}\left(x_1,x_2,\frac{Q^2}{m_b^2},\alpha_s^{(4)}\left(Q^2\right)\right)\\ +&\int dx_1dx_2\,\left[f_{\bar{b}}^{(4)}\left(x_1\right)f_g^{(4)}\left(x_2,Q^2\right)+\left(x_1\to x_2\right)\right]\hat{\sigma}_{\bar{b}g}^{(4)}\left(x_1,x_2,\frac{Q^2}{m_b^2},\alpha_s^{(4)}\left(Q^2\right)\right).\end{aligned} \tag{92}$$

The Feynman diagrams for these new contributions are shown in Fig. (5).

Therefore at order α_s the generalized FONLL prescription reads

$$\sigma^{FONLL}=\sigma^{FNRL}+\Delta\sigma, \tag{93}$$

where σ^{FNRL} is the result of the standard FONLL method given in the previous

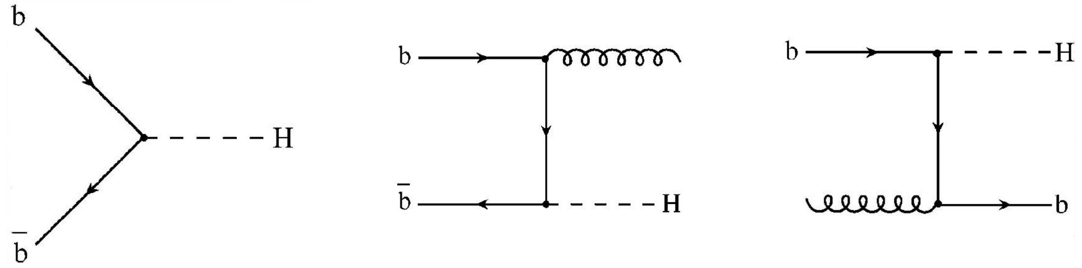

Figure 5: From left to right, leading-order and next-to-leading new contributions to the four flavour scheme hard cross sections. They are the same diagrams of Fig. (4), but the computation has now to be performed retaining the complete mass dependence.

section and

$$\Delta\sigma = \Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3,$$

$$\begin{aligned}
\Delta\sigma_1 &= \int dx_1 dx_2 \, \left[f_b^{(4)}(x_1) f_{\bar{b}}^{(4)}(x_2) + (x_1 \leftrightarrow x_2) \right] \left[\hat{\sigma}_{b\bar{b}}^{(4)} - \hat{\sigma}_{b\bar{b}}^{(4,0)} \right] \\
\Delta\sigma_2 &= \int dx_1 dx_2 \, \left[f_b^{(4)}(x_1) f_g^{(4)}\left(x_2, Q^2\right) + (x_1 \leftrightarrow x_2) \right] \left[\hat{\sigma}_{bg}^{(4)} - \hat{\sigma}_{bg}^{(4,0)} \right] \\
\Delta\sigma_3 &= \int dx_1 dx_2 \, \left[f_{\bar{b}}^{(4)}(x_1) f_g^{(4)}\left(x_2, Q^2\right) + (x_1 \leftrightarrow x_2) \right] \left[\hat{\sigma}_{\bar{b}g}^{(4)} - \hat{\sigma}_{\bar{b}g}^{(4,0)} \right].
\end{aligned} \tag{94}$$

In order to complete the construction of the FONLL method, we have to express both $\alpha_s^{(4)}$ and $f_i^{(4)}$ in terms of $\alpha_s^{(5)}$ and $f_i^{(5)}$. As already seen in the previous sections, this is achieved through suitable matching conditions between the two schemes at a certain initial scale m^2, and then through evolution up to a generic scale Q^2 through DGLAP equations. As for matching conditions for the strong coupling, Eq. (17) still holds,

$$\alpha_s^{(5)}\left(Q^2\right) = \alpha_s^{(4)}\left(Q^2\right) + \frac{2T_R}{3}\frac{L}{2\pi}\alpha_s^2\left(m^2\right) + O\left(\alpha_s^3\right).$$

However, concerning matching PDFs, since we are considering the presence of heavy quarks in the initial state we have to use the conditions already discussed in the contest of DIS with intrinsic charm. These include new coefficients K_{ib} and $K_{i\bar{b}}$ starting at $O(\alpha_s)$, given in Eq. (42). The matching functions at the initial scale m^2 therefore read, omitting explicit x dependence

$$f_i^{(5)}\left(m^2\right) = \sum_j K_{ij}\left(m^2\right) \otimes f_j^{(4)}\left(m^2\right),$$

with $i, j = q, \bar{q}, g, b, \bar{b}$ and $K_{ij}\left(m^2\right) = \sum_n \alpha_s^n K_{ij}^{(n)}$. Evolving them up to $O(\alpha_s)$

and remembering that the heavy quark PDFs in the 4FS do not evolve, we get

$$
\begin{aligned}
f_b^{(4)} =& f_b^{(5)}\left(Q^2\right) - \alpha_s^{(4)}\left(Q^2\right) K_{bb}^{(1)} \otimes f_b^{(4)} \\
& - \alpha_s^{(5)}\left(Q^2\right) P_{qq}^{(0)} L \otimes f_b^{(5)}\left(Q^2\right) \\
& - \alpha_s^{(5)}\left(Q^2\right) P_{qg}^{(0)} L \otimes f_g^{(5)}\left(Q^2\right) \\
=& f_b^{(5)}\left(Q^2\right) - \alpha_s^{(5)}\left(Q^2\right)\left[K_{bb}^{(1)} + P_{qq}^{(0)} L\right] \otimes f_b^{(5)}\left(Q^2\right) \\
& - \alpha_s^{(5)}\left(Q^2\right) P_{qg}^{(0)} L \otimes f_g^{(5)}\left(Q^2\right).
\end{aligned} \tag{95}
$$

In the same way, for the PDF of $\bar{b}$

$$
\begin{aligned}
f_{\bar{b}}^{(4)} =& f_{\bar{b}}^{(5)}\left(Q^2\right) - \alpha_s^{(5)}\left(Q^2\right)\left[K_{bb}^{(1)} + P_{qq}^{(0)} L\right] \otimes f_{\bar{b}}^{(5)}\left(Q^2\right) \\
& - \alpha_s^{(5)}\left(Q^2\right) P_{qg}^{(0)} L \otimes f_g^{(5)}\left(Q^2\right).
\end{aligned} \tag{96}
$$

In all the PDFs of the equations above the x dependence is understood. In the contest of generalized FONLL applied to DIS the gluon PDF doesn't appear in the corrections terms. However in the hadronic case the correction terms of Eq. (94) do contain the gluon PDF, so we have to work out its expression in terms of 5FS quantities. Eq. (23) does not hold anymore, because of the presence of new coefficients of order α_s. The general expression for the matching condition of the gluon PDF at a initial scale m_b^2 now is given by

$$
\begin{aligned}
& f_g^{(5)}\left(x, m_b^2\right) = \\
& \quad f_g^{(4)}\left(x, m_b^2\right) + \alpha_s \int_0^1 \frac{dy}{y} \sum_{i=q,\bar{q},b,\bar{b},g} K_{gi}^{(1)}(y)\, f_i^{(4)}\left(\frac{x}{y}, m_b^2\right) + O\left(\alpha_s^2\right) \\
& \quad = f_g^{(4)}\left(x, m_b^2\right) + \alpha_s \sum_{i=b,\bar{b}} K_{gi}^{(1)} \otimes f_i^{(4)}\left(m_b^2\right) + O\left(\alpha_s^2\right).
\end{aligned} \tag{97}
$$

Evolving this equation with Altarelli Parisi up to scale Q^2 we get, performing the same calculations done in getting eq.(23) and omitting the explicit dependence from x,

$$
\begin{aligned}
& f_g^{(4)}\left(Q^2\right) - \alpha_s^{(4)}\left(Q^2\right) P_{gi}^{(4),(0)} L \otimes f_i^{(4)}\left(Q^2\right) \\
& \qquad - \alpha_s^{(4)}\left(Q^2\right) P_{gg}^{(4),(0)} L \otimes f_g^{(4)}\left(Q^2\right) \\
& = f_g^{(5)}\left(Q^2\right) - \alpha_s^{(5)}\left(Q^2\right) P_{gi}^{(5),(0)} L \otimes f_i^{(5)}\left(Q^2\right) \\
& \qquad - \alpha_s^{(5)}\left(Q^2\right) P_{gg}^{(5),(0)} L \otimes f_g^{(5)}\left(Q^2\right) \\
& \qquad - \alpha_s^{(4)}\left(Q^2\right) K_{gb}^{(1)} \otimes f_b^{(4)}.
\end{aligned} \tag{98}
$$

Therefore, at order α_s, we have, reintroducing the explicit dependence on x

$$\begin{aligned}
f_g^{(4)}\left(x,Q^2\right) = f_g^{(5)}\left(x,Q^2\right) &+ \alpha_s^{(5)}\left(Q^2\right) L\left[P_{gi}^{(4),(0)} - P_{gi}^{(5),(0)}\right] \otimes f_i^{(5)}\left(x,Q^2\right) \\
&+ \alpha_s^{(5)}\left(Q^2\right) L\left[P_{gg}^{(4),(0)} - P_{gg}^{(5),(0)}\right] \otimes f_g^{(5)}\left(x,Q^2\right) \\
&- \alpha_s^{(5)}\left(Q^2\right) K_{gb}^{(1)} \otimes f_b^{(5)} = \\
&f_g^{(5)}\left(x,Q^2\right) + \alpha_s^{(5)}\left(Q^2\right) L \frac{2T_R}{3\left(2\pi\right)} \int_x^1 \frac{dy}{y} \delta\left(1-y\right) f_g^{(5)}\left(\frac{x}{y},Q^2\right) \\
&- \alpha_s^{(5)}\left(Q^2\right) K_{gb}^{(1)} \otimes f_b^{(5)}\left(x,Q^2\right),
\end{aligned} \tag{99}$$

where in the last passagge we used

$$P_{ij}^{(5),(0)}\left(z\right) - P_{ij}^{(4),(0)}\left(z\right) = -\delta_{ig}\delta_{jg} \frac{2T_R}{\left(2\pi\right)3} \delta\left(1-z\right) \tag{100}$$

So finally we obtain

$$\begin{aligned}
f_g^{(4)}\left(x,Q^2\right) = f_g^{(5)}\left(x,Q^2\right) &+ \alpha_s^{(5)}\left(Q^2\right) L \frac{2T_R}{3\left(2\pi\right)} f_g^{(5)}\left(x,Q^2\right) \\
&- \alpha_s^{(5)}\left(Q^2\right) K_{gb}^{(1)} \otimes f_b^{(5)}\left(x,Q^2\right) + O\left(\alpha_s^2\right)
\end{aligned} \tag{101}$$

Substituting Eq. (95) and (101) in Eq. (94) we get (for simplicity we omit the dependency from x_1, x_2, m_b, which will be reinserted in the final result)

$$\begin{aligned}
\Delta\sigma_1 = &\int dx_1 dx_2 \\
&\left\{f_b^{(5)}\left(Q^2\right) - \alpha_s^{(5)}\left(Q^2\right)\left[K_{bb}^{(1)} + P_{qq}^{(0)} L\right] \otimes f_b^{(5)}\left(Q^2\right) - \alpha_s^{(5)}\left(Q^2\right) P_{qg}^{(0)} L \otimes f_g^{(5)}\left(Q^2\right)\right\} \\
&\left\{f_{\bar b}^{(5)}\left(Q^2\right) - \alpha_s^{(5)}\left(Q^2\right)\left[K_{bb}^{(1)} + P_{qq}^{(0)} L\right] \otimes f_{\bar b}^{(5)}\left(Q^2\right) - \alpha_s^{(5)}\left(Q^2\right) P_{qg}^{(0)} L \otimes f_g^{(5)}\left(Q^2\right)\right\} \\
&\left[\left(\hat\sigma_{b\bar b}^{(4),0} - \hat\sigma_{b\bar b}^{(4,0),0}\right) + \left(\hat\sigma_{b\bar b}^{(4),1} - \hat\sigma_{b\bar b}^{(4,0),1}\right) \alpha_s^{(4)}\left(Q^2\right)\right] \\
&+ \int dx_1 dx_2 \left\{\left(x_1 \leftrightarrow x_2\right)\right\} \left\{\left(x_1 \leftrightarrow x_2\right)\right\} \\
&\left[\left(\hat\sigma_{b\bar b}^{(4),0} - \hat\sigma_{b\bar b}^{(4,0),0}\right) + \left(\hat\sigma_{b\bar b}^{(4),1} - \hat\sigma_{b\bar b}^{(4,0),1}\right) \alpha_s^{(4)}\left(Q^2\right)\right] \\
&= \int dx_1 dx_2 \left\{\Delta\sigma_1^{(0)} + \Delta\sigma_1^{(1)} \alpha_s^{(5)}\left(Q^2\right) + O\left(\alpha_s^2\right)\right\},
\end{aligned} \tag{102}$$

with

$$\Delta\sigma_1^{(0)} = \left[f_b^{(5)}\left(Q^2\right) f_{\bar{b}}^{(5)}\left(Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right]\left[\hat{\sigma}_{b\bar{b}}^{(4),0} - \hat{\sigma}_{b\bar{b}}^{(4,0),0}\right]$$

$$\begin{aligned}\Delta\sigma_1^{(1)} &= \left[f_b^{(5)}\left(Q^2\right) f_{\bar{b}}^{(5)}\left(Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right]\left[\hat{\sigma}_{b\bar{b}}^{(4),1} - \hat{\sigma}_{b\bar{b}}^{(4,0),1}\right] \\ &+ \left\{\left(-\left(K_{bb}^{(1)} + P_{qq}^{(0)} L\right) \otimes f_{\bar{b}}^{(5)}\left(Q^2\right) - P_{qg}^{(0)} L \otimes f_g^{(5)}\left(Q^2\right)\right) f_b^{(5)}\left(Q^2\right)\right. \\ &+ \left.\left(b \leftrightarrow \bar{b}, x_1 \leftrightarrow x_2\right) + \left(x_1 \leftrightarrow x_2\right)\right\}\left[\hat{\sigma}_{b\bar{b}}^{(4),0} - \hat{\sigma}_{b\bar{b}}^{(4,0),0}\right];\end{aligned} \tag{103}$$

$$\begin{aligned}\Delta\sigma_2 = &\int dx_1 dx_2 \\ &\left\{f_b^{(5)}\left(Q^2\right) - \alpha_s^{(5)}\left(Q^2\right)\left[K_{bb}^{(1)} + P_{qq}^{(0)} L\right] \otimes f_b^{(5)}\left(Q^2\right) - \alpha_s^{(5)}\left(Q^2\right) P_{qg}^{(0)} L \otimes f_g^{(5)}\left(Q^2\right)\right\} \\ &\left\{f_g^{(5)}\left(Q^2\right) - \alpha_s^{(5)}\left(Q^2\right) L \frac{2T_R}{3} f_g^{(5)}\left(Q^2\right) - \alpha_s^{(5)}\left(Q^2\right) K_{gb}^{(1)} \otimes f_b^{(5)}\left(Q^2\right)\right\} \\ &\left[\left(\hat{\sigma}_{bg}^{(4),1} - \hat{\sigma}_{bg}^{(4,0),1}\right)\alpha_s^{(4)}\left(Q^2\right)\right] \\ &+ \int dx_1 dx_2 \left\{\left(x_1 \leftrightarrow x_2\right)\right\}\left\{\left(x_1 \leftrightarrow x_2\right)\right\}\left[\left(\hat{\sigma}_{bg}^{(4),1} - \hat{\sigma}_{bg}^{(4,0),1}\right)\alpha_s^{(4)}\left(Q^2\right)\right] \\ &= \int dx_1 dx_2 \left[f_b^{(5)}\left(Q^2\right) f_g^{(5)}\left(Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right]\left[\hat{\sigma}_{bg}^{(4),1} - \hat{\sigma}_{bg}^{(4,0),1}\right]\alpha_s^{(5)}\left(Q^2\right).\end{aligned} \tag{104}$$

In the same way we get

$$\Delta\sigma_3 = \int dx_1 dx_2 \left[f_{\bar{b}}^{(5)}\left(Q^2\right) f_g^{(5)}\left(Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right]\left[\hat{\sigma}_{\bar{b}g}^{(4),1} - \hat{\sigma}_{\bar{b}g}^{(4,0),1}\right]\alpha_s^{(5)}\left(Q^2\right). \tag{105}$$

Thus we have obtained expressions for the correction terms written as function of the 5FS PDFs, the massive partonic cross sections, their massless limits and the matching coefficients. We notice that, unlike the standard FONLL case where the results started being different from the 5FS only at order $O\left(\alpha_s^2\right)$, considering initial-state heavy quarks also in the 4FS, the FONLL scheme differs from 5FS already at order α_s^0. In fact we have

$$\begin{aligned}\sigma^{FONLL-A,(0)} = \int dx_1 dx_2 &\left[f_{\bar{b}}^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \\ &\times\left[\hat{\sigma}_{b\bar{b}}^{(5),0} + \hat{\sigma}_{b\bar{b}}^{(4),0} - \hat{\sigma}_{b\bar{b}}^{(4,0),0}\right],\end{aligned} \tag{106}$$

$$
\begin{aligned}
&\sigma^{FONLL-A,(1)} = \int dx_1 dx_2 \\
&\left[f_{\bar{b}}^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{b\bar{b}}^{(5),1} \\
&+ \left[f_g^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) + \left(x_1 \rightarrow x_2\right)\right] \hat{\sigma}_{gb}^{(5),1} + \left(b \rightarrow \bar{b}\right) \\
&+ \Delta\sigma_1^{(1)} + \Delta\sigma_2 + \Delta\sigma_3,
\end{aligned}
\tag{107}
$$

thus using explicit expressions above we get

$$
\begin{aligned}
&\sigma^{FONLL-A,(1)} = \int dx_1 dx_2 \\
&\left[f_{\bar{b}}^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right]\left[\hat{\sigma}_{b\bar{b}}^{(5),1} + \hat{\sigma}_{b\bar{b}}^{(4),1} - \hat{\sigma}_{b\bar{b}}^{(4,0),1}\right] \\
&+ \left\{\left(-\left(K_{bb}^{(1)} + P_{qq}^{(0)} L\right) \otimes f_{\bar{b}}^{(5)}\left(x_2, Q^2\right) - P_{qg}^{(0)} L \otimes f_g^{(5)}\left(x_2, Q^2\right)\right) f_b^{(5)}\left(x_1, Q^2\right)\right. \\
&\left. + \left(b \leftrightarrow \bar{b}, x_1 \leftrightarrow x_2\right) + \left(x_1 \leftrightarrow x_2\right)\right\}\left[\hat{\sigma}_{b\bar{b}}^{(4),0} - \hat{\sigma}_{b\bar{b}}^{(4,0),0}\right] \\
&+ \left[f_b^{(5)}\left(x_1, Q^2\right) f_g^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right]\left[\hat{\sigma}_{gb}^{(5),1} + \hat{\sigma}_{bg}^{(4),1} - \hat{\sigma}_{bg}^{(4,0),1}\right] + \left(b \rightarrow \bar{b}\right),
\end{aligned}
\tag{108}
$$

where we have used the results of the previous section. To avoid confusion, note that in the equation above, by, for example, the expression $P_{qq}^{(0)} L \otimes f_{\bar{b}}^{(5)}\left(x, Q^2\right)$ we mean $L \int_x^1 \frac{dy}{y} P_{qq}^{(0)}\left(\frac{x}{y}\right) f_{\bar{b}}^{(5)}\left(y, Q^2\right)$.

In Eq. (106) and (108) are present both massive partonic cross sections $\sigma^{(4)}$ and their massless limit $\sigma^{(4,0)}$. In order to obtain simpler expressions which allow to highlight some general features of the results, it could be useful to express the massless limit of the 4FS cross sections in terms of the 5FS ones, as previously done in the case of DIS in Eq. (48). Therefore we will now work out relations between massless limit of the 4FS and the 5FS cross sections, in order to simplify the expressions above. We start from equation at scale m^2

$$
\begin{aligned}
&\int dx_1 dx_2\, f_b^{(4)}\left(x_1\right) f_{\bar{b}}^{(4)}\left(x_2\right) \hat{\sigma}_{b\bar{b}}^{(4,0)}\left(x_1, x_2, m^2, \alpha_s^{(4)}\left(m^2\right)\right) \\
&= \int dx_1 dx_2\, f_b^{(5)}\left(x_1, m^2\right) f_{\bar{b}}^{(5)}\left(x_2, m^2\right) \hat{\sigma}_{b\bar{b}}^{(5)}\left(x_1, x_2, \alpha_s^{(5)}\left(m^2\right)\right)
\end{aligned}
\tag{109}
$$

which has to be satisfied order by order in the same coupling (as already observed several times, this expresses the fact that the double counted term in FONLL can be obtained both as massless limit of the massive scheme and as fixed order expansion of the massless scheme). Evolving the previous equation to scale Q^2

the PDFs on left hand side remains the same, while on the right hand side we used Eq. (95) so we get

$$\begin{aligned}
&\int dx_1 dx_2\, f_b^{(4)}(x_1)\, f_{\bar{b}}^{(4)}(x_2)\, \hat{\sigma}_{b\bar{b}}^{(4,0)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(4)}\left(Q^2\right)\right) \\
&= \int dx_1 dx_2 \\
&\left[f_b^{(4)}(x_1) + \alpha_s^{(5)}(x_1)\left[K_{bb}^{(1)} + P_{qq}^{(0)} L\right] \otimes f_b^{(4)}(x_1) + \alpha_s^{(4)}\left(Q^2\right) P_{qg}^{(0)} L \otimes f_g^{(4)}\left(x_1, Q^2\right)\right] \\
&\left[f_{\bar{b}}^{(4)}(x_2) + \alpha_s^{(5)}\left(Q^2\right)\left[K_{bb}^{(1)} + P_{qq}^{(0)} L\right] \otimes f_{\bar{b}}^{(4)}(x_2) + \alpha_s^{(4)}\left(Q^2\right) P_{qg}^{(0)} L \otimes f_g^{(4)}\left(x_2, Q^2\right)\right] \\
&\left\{\hat{\sigma}_{b\bar{b}}^{(5),0}\left(x_1, x_2, \alpha_s^{(5)}\left(Q^2\right)\right) + \alpha_s^{(5)}\left(Q^2\right) \hat{\sigma}_{b\bar{b}}^{(5),1}\left(x_1, x_2, \alpha_s^{(5)}\left(Q^2\right)\right) + O\left(\alpha_s^2\right)\right\},
\end{aligned} \tag{110}$$

from which we read, omitting the dependence on the scale of the process

$$\hat{\sigma}_{b\bar{b}}^{(4,0),0}(x_1, x_2) = \hat{\sigma}_{b\bar{b}}^{(5),0}(x_1, x_2), \tag{111}$$

and

$$\begin{aligned}
&\int dx_1 dx_2\, f_b^{(4)}(x_1)\, f_{\bar{b}}^{(4)}(x_2)\, \hat{\sigma}_{b\bar{b}}^{(4,0),1}(x_1, x_2) \\
&= \int dx_1 dx_2 f_b^{(4)}(x_1)\, f_{\bar{b}}^{(4)}(x_2)\, \hat{\sigma}_{b\bar{b}}^{(5),1}(x_1, x_2) \\
&+ \int dx_1 dx_2 f_b^{(4)}(x_1)\, f_{\bar{b}}^{(4)}(x_2) \int_0^1 dy \left[K_{bb}^{(1)}(y) + P_{qq}^{(0)} L\right] \\
&\qquad \times \left[\hat{\sigma}_{b\bar{b}}^{(5),0}(x_1, y x_2) + \hat{\sigma}_{b\bar{b}}^{(5),0}(y x_1, x_2)\right] \\
&+ \int dx_1 dx_2 f_{\bar{b}}^{(4)}(x_2)\, f_g^{(4)}\left(x_1, Q^2\right) L \int_0^1 dy P_{qq}^{(0)}(y)\, \hat{\sigma}_{b\bar{b}}^{(5),0}(y x_1, x_2) \\
&+ \int dx_1 dx_2 f_b^{(4)}(x_1)\, f_g^{(4)}\left(x_2, Q^2\right) L \int_0^1 dy P_{qq}^{(0)}(y)\, \hat{\sigma}_{b\bar{b}}^{(5),0}(x_1, y x_2),
\end{aligned} \tag{112}$$

where in both right and left side of Eq. (112) we can replace $f_i^{(4)}$ with $f_i^{(5)}$, since their difference is at least of order α_s, and the equality is between quantities which are already multiplied by α_s.

Similarly, starting from

$$\begin{aligned}
&\int dx_1 dx_2\, f_b^{(4)}(x_1)\, f_g^{(4)}\left(x_2, m^2\right) \hat{\sigma}_{bg}^{(4,0)}(x_1, x_2) \\
&= \int dx_1 dx_2\, f_b^{(5)}\left(x_1, m^2\right) f_g^{(5)}\left(x_2, m^2\right) \hat{\sigma}_{bg}^{(5)}(x_1, x_2),
\end{aligned} \tag{113}$$

evolving to a scale Q^2, we get

$$\hat{\sigma}_{bg}^{(4,0),1}(x_1, x_2) = \hat{\sigma}_{bg}^{(5),1}(x_1, x_2). \tag{114}$$

Replacing Eq. (111), (112) and (114) in Eq. (106) and (108) we get the final expression for the generalized FONLL up to order α_s, expressed in terms of only the massive partonic cross sections. As already noticed, the power expansion is referred to the perturbative series of the partonic cross section.

$$\begin{aligned}\sigma^{FONLL-A,(0)} = &\int dx_1 dx_2 \\ &\left[f_{\bar{b}}^{(5)}\left(x_1,Q^2\right) f_b^{(5)}\left(x_2,Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{b\bar{b}}^{(4),0}\left(x_1,x_2\right)\end{aligned} \tag{115}$$

$$\begin{aligned}\sigma^{FONLL-A,(1)} = &\int dx_1 dx_2 \\ &\left[f_{\bar{b}}^{(5)}\left(x_1,Q^2\right) f_b^{(5)}\left(x_2,Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{b\bar{b}}^{(4),1} \\ &+ \left[f_b^{(5)}\left(x_1,Q^2\right) f_g^{(5)}\left(x_2,Q^2\right) + \left(x_1 \leftrightarrow x_2\right) + \left(b \to \bar{b}\right)\right] \hat{\sigma}_{bg}^{(4),1} \\ &+ \left\{\left(-\left(K_{bb}^{(1)} + P_{qq}^{(0)} L\right) \otimes f_{\bar{b}}^{(5)}\left(x_2,Q^2\right) - P_{qg}^{(0)} L \otimes f_g^{(5)}\left(x_2,Q^2\right)\right) f_b^{(5)}\left(x_1,Q^2\right)\right. \\ &\left.+ \left(b \leftrightarrow \bar{b}, x_1 \leftrightarrow x_2\right) + \left(x_1 \leftrightarrow x_2\right)\right\} \hat{\sigma}_{b\bar{b}}^{(4),0}.\end{aligned} \tag{116}$$

This is our main result. A more explicit form of Eq. (116) is given by

$$\begin{aligned}\sigma^{FONLL-A,(1)} = &\int dx_1 dx_2 \\ &\left[f_{\bar{b}}^{(5)}\left(x_1,Q^2\right) f_b^{(5)}\left(x_2,Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{b\bar{b}}^{(4),1}\left(x_1,x_2\right) \\ &+ \left[f_b^{(5)}\left(x_1,Q^2\right) f_g^{(5)}\left(x_2,Q^2\right) + \left(x_1 \leftrightarrow x_2\right) + \left(b \to \bar{b}\right)\right] \hat{\sigma}_{bg}^{(4),1}\left(x_1,x_2\right) \\ &- \int dx_1 dx_2 \left\{f_b^{(5)}\left(x_1,Q^2\right) f_{\bar{b}}^{(5)}\left(x_2,Q^2\right) \int_0^1 dy \left[K_{bb}^{(1)}\left(y\right) + P_{qq}^{(0)} L\right] + \left(x_1 \leftrightarrow x_2\right)\right\} \\ &\qquad \times \left[\hat{\sigma}_{b\bar{b}}^{(5),0}\left(x_1, y x_2\right) + \hat{\sigma}_{b\bar{b}}^{(5),0}\left(y x_1, x_2\right)\right] \\ &- \int dx_1 dx_2 \left\{f_{\bar{b}}^{(5)}\left(x_2,Q^2\right) f_g^{(5)}\left(x_1,Q^2\right) L \int_0^1 dy P_{qg}^{(0)}\left(y\right) + \left(x_1 \leftrightarrow x_2\right)\right\} \hat{\sigma}_{b\bar{b}}^{(4),0}\left(y x_1, x_2\right) \\ &- \int dx_1 dx_2 \left\{f_b^{(5)}\left(x_1,Q^2\right) f_g^{(5)}\left(x_2,Q^2\right) L \int_0^1 dy P_{qg}^{(0)}\left(y\right) + \left(x_1 \leftrightarrow x_2\right)\right\} \hat{\sigma}_{b\bar{b}}^{(4),0}\left(x_1, y x_2\right).\end{aligned} \tag{117}$$

We notice that Eqs. (115) and (116) are expressions for the total cross section obtained using the massless 5FS PDFs convoluted with massive cross section, from which we subtract terms containing the unresummed collinear logarithms, which are terms proportional to $P_{qg}^{(0)} L \otimes f_g^{(5)}(Q^2)$ and $P_{qq}^{(0)} L \otimes f_{\bar{b}}^{(5)}(Q^2)$, coming from

$$\begin{aligned} f_b\left(x, Q^2\right) =& \alpha_s\left(Q^2\right) L \int_x^1 \frac{dy}{y} P_{qg}\left(\frac{x}{y}\right) f_g\left(y, Q^2\right) \\ &+ \alpha_s\left(Q^2\right) L \int_x^1 \frac{dy}{y} P_{qq}\left(\frac{x}{y}\right) f_b\left(y, Q^2\right). \end{aligned} \tag{118}$$

In addition to this, the new coefficient $K_{b\bar{b}}^{(1)}$ appears, accounting for the presence of initial-state heavy quarks also in the 4FS. Therefore Eq. (116) can be rewritten as

$$\begin{aligned} &\sigma^{FONLL-A,(1)} = \int dx_1 dx_2 \\ &\left[f_{\bar{b}}^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\hat{\sigma}}_{b\bar{b}}^{(4),1} \\ &+ \left[f_b^{(5)}\left(x_1, Q^2\right) f_g^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right) + \left(b \rightarrow \bar{b}\right)\right] \hat{\hat{\sigma}}_{bg}^{(4),1} \\ &+ \left\{-\left(K_{bb}^{(1)} \otimes f_{\bar{b}}^{(5)}\left(x_2, Q^2\right)\right) f_b^{(5)}\left(x_1, Q^2\right) + \left(b \leftrightarrow \bar{b}, x_1 \leftrightarrow x_2\right) + \left(x_1 \leftrightarrow x_2\right)\right\} \hat{\sigma}_{b\bar{b}}^{(4),0} \end{aligned} \tag{119}$$

where $\hat{\hat{\sigma}}_{b\bar{b}}^{(4),1}$ and $\hat{\hat{\sigma}}_{bg}^{(4),1}$ are the massive 4FS cross sections from which the large logarithms due to the collinear emissions of heavy quark have been removed.

This fact can be stated also by saying that the difference terms defined in the previous section $\sigma^{(d)}$ is such that $\sigma^{(d),0}$ $\sigma^{(d),1}$ vanish identically, without expanding heavy quark PDFs, just as in the DIS case. This can be explicitly seen by writing the difference terms at order $O\left(\alpha_s^0\right)$ and $O\left(\alpha_s\right)$:

$$\begin{aligned} \sigma^{(d),0} =& \int dx_1 dx_2 \left[f_{\bar{b}}^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{b\bar{b}}^{(5),0} \\ &- \int dx_1 dx_2 \left[f_{\bar{b}}^{(4)}\left(x_1\right) f_b^{(4)}\left(x_2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{b\bar{b}}^{(4,0),0} \end{aligned} \tag{120}$$

$$\begin{aligned} \sigma^{(d),1} =& \int dx_1 dx_2 \Big\{\left[f_{\bar{b}}^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{b\bar{b}}^{(5),1} \\ &+ \left[f_g^{(5)}\left(x_1, Q^2\right) f_b^{(5)}\left(x_2, Q^2\right) + \left(x_1 \rightarrow x_2\right)\right] \hat{\sigma}_{gb}^{(5),1}\Big\} + \left(b \rightarrow \bar{b}\right) \\ &- \int dx_1 dx_2 \Big\{\left[f_{\bar{b}}^{(4)}\left(x_1\right) f_b^{(4)}\left(x_2\right) + \left(x_1 \leftrightarrow x_2\right)\right] \hat{\sigma}_{b\bar{b}}^{(4,0),1} \\ &+ \left[f_g^{(4)}\left(x_1, Q^2\right) f_b^{(4)}\left(x_2\right) + \left(x_1 \rightarrow x_2\right)\right] \hat{\sigma}_{gb}^{(4,0),1}\Big\} - \left(b \rightarrow \bar{b}\right). \end{aligned} \tag{121}$$

Using matching conditions Eq. (95) and(101) together with Eq. (111), (112) and (114), we find that these terms identically vanish, without expanding the bottom quark PDFs.

These same results, here derived in the contest of the FONLL method, were obtained through a different procedure in Ref. [9]. In Section 2 of Ref. [9], it is suggested that, in order to obtain accurate results on the total cross section for Higgs production in bottom quarks fusion, one should consider a theoretically-defined heavy-quark distribution function, just as it is usually done in the massless scheme, and use the DGLAP equations to sum all the collinear logarithms. The PDF set obtained in this way should then be convoluted with the massive partonic cross sections. The latter will contain unresummed large logarithms coming from the region in which a gluon splits in a nearly collinear $h, \bar{h}$ pair, but these terms have already been summed into the heavy quark PDF, and therefore have to be subtracted. This is exactly what we have just verified using FONLL scheme. Eq. (3), (4) and (15) of Ref. [9] give the total cross section up to order α_s obtained following these considerations, and comparing them with Eq. (115) and (116) we find out they are exactly the same. Particularly we have that Eq. (115) corresponds to Eq. (3) of Ref. [9], while the Eq. (116) correspond to the sum between Eq. (4) and eq.(15) of Ref. [9]. The only difference is that in our results the new term $K^{(1)}_{b\bar{b}}$ appears, which takes into account the presence of initial state heavy quarks also in the 4FS. Therefore we have obtained through the generalized FONLL scheme the same results of Ref. [9].

A final observation regarding the results of this subsection is that, even if we consider $f^{(5)}_b\left(x, m_b^2\right) = f^{(5)}_{\bar{b}}\left(x, m_b^2\right) = 0$, the correction terms of Eq. (94), though subleading, do not vanish when $Q^2 > m_b^2$. They only vanish for $Q^2 = m_b^2$. This same observation may well be done for the case of DIS, and the reason is that, when we re-express $f^{(4)}_i$ in term of $f^{(5)}_i$ at scale Q^2 through evolved matching conditions, only terms up to $O\left(\alpha_s\right)$ are kept. Therefore, in other words, even if we consider the intrinsic heavy quark to be zero, we don't find exactly the same results we would get using standard FONLL. The first thing we will have to check studying the phenomenology is that this difference is indeed negligible, as already checked for the case of DIS in Ref. [5].

To sum up, in this subsection we have derived results for FONLL method up to order α_s, accounting for initial state heavy quarks also in the 4FS. We have found that our results coincide with those described by Ref.[9]. However, while in Ref. [9] the results were not obtained in the context of a well-defined factorization scheme, here we have obtained results through the generalized FONLL method, working in well-defined factorization schemes and giving a general and solid theory of the topic. In addition to this our formalism allow to explicitly account for initial-state heavy quarks also in the massive scheme

5.3 Order α_s^2

In this subsection we discuss the main steps which have to be followed in order to obtain the generalized FONLL scheme up to order $O\left(\alpha_s^2\right)$, which is required to fully generalized the results of the previous section. In order to do this, we will first have to identify the relevant subprocesses, which are more than the previous case, since we have to consider also partonic processes starting at $O\left(\alpha_s^2\right)$. Then matching conditions up to $O\left(\alpha_s^2\right)$ are required. As a final cross-check, we should also express the massless limit of 4FS cross section in term of 5FS cross section, in order to verify if the same general results of the previous subsection still work at higher order, but this is not done here.

Firstly we start writing down the relevant partonic subprocesses up to order α_s^2. These are the same we wrote for the 5FS, but now the full mass dependence has to be retained. Therefore we have the subprocesses:

- up to two loops: $b\bar{b} \to h$ fig.(5)
- up to one loop: $b\bar{b} \to hg$, $bg \to bh$ fig.(5)
- at tree level: $b\bar{b} \to ggh$, $b\bar{b} \to q\bar{q}h$, $b\bar{b} \to b\bar{b}h$, $bg \to bgh$, $bq \to hbq$, $bb \to hb\bar{b}$, $q\bar{q} \to hb\bar{b}$, $gg \to hb\bar{b}$ fig.(3), (6)

and the relevant orders are

$$\hat{\sigma}_{b\bar{b}}^{(4)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(5)}\left(Q^2\right)\right) = \hat{\sigma}_{b\bar{b}}^{(4),0}\left(x_1, x_2, \frac{Q^2}{m^2}\right) + \alpha_s^{(5)}\left(Q^2\right)\hat{\sigma}_{b\bar{b}}^{(4),1}\left(x_1, x_2, \frac{Q^2}{m^2}\right) + \left(\alpha_s^{(4)}\left(Q^2\right)\right)^2\hat{\sigma}_{b\bar{b}}^{(4),2}\left(x_1, x_2, \frac{Q^2}{m^2}\right) + O\left(\alpha_s^3\right), \tag{122}$$

$$\hat{\sigma}_{bg}^{(4)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(4)}\left(Q^2\right)\right) = \alpha_s^{(4)}\left(Q^2\right)\hat{\sigma}_{bg}^{(4),1}\left(x_1, x_2, \frac{Q^2}{m^2}\right) + \left(\alpha_s^{(4)}\left(Q^2\right)\right)^2\hat{\sigma}_{bg}^{(4),2}\left(x_1, x_2, \frac{Q^2}{m^2}\right) + O\left(\alpha_s^3\right), \tag{123}$$

$$\hat{\sigma}_{bq}^{(4)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(5)}\left(Q^2\right)\right) = \left(\alpha_s^{(4)}\left(Q^2\right)\right)^2\hat{\sigma}_{bq}^{(4),2}\left(x_1, x_2, \frac{Q^2}{m^2}\right) + O\left(\alpha_s^3\right), \tag{124}$$

$$\hat{\sigma}_{gg}^{(4)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(4)}\left(Q^2\right)\right) = \left(\alpha_s^{(4)}\left(Q^2\right)\right)^2\hat{\sigma}_{gg}^{(4),2}\left(x_1, x_2, \frac{Q^2}{m^2}\right) + O\left(\alpha_s^3\right), \tag{125}$$

$$\hat{\sigma}_{bb}^{(4)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(4)}\left(Q^2\right)\right) = \left(\alpha_s^{(4)}\left(Q^2\right)\right)^2\hat{\sigma}_{bb}^{(4),2}\left(x_1, x_2, \frac{Q^2}{m^2}\right) + O\left(\alpha_s^3\right), \tag{126}$$

$$\hat{\sigma}^{(4)}_{q\bar{q}}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(4)}\left(Q^2\right)\right) = \left(\alpha_s^{(4)}\left(Q^2\right)\right)^2 \hat{\sigma}^{(4),2}_{q\bar{q}}\left(x_1, x_2, \frac{Q^2}{m^2}\right) + O\left(\alpha_s^3\right), \quad (127)$$

The diagrams in Fig. (5) up to order α_s^2 take 2-loops and 1-loops corrections respectively, while the new tree-level processes we have to consider are depicted in the diagrams of Fig. (6).

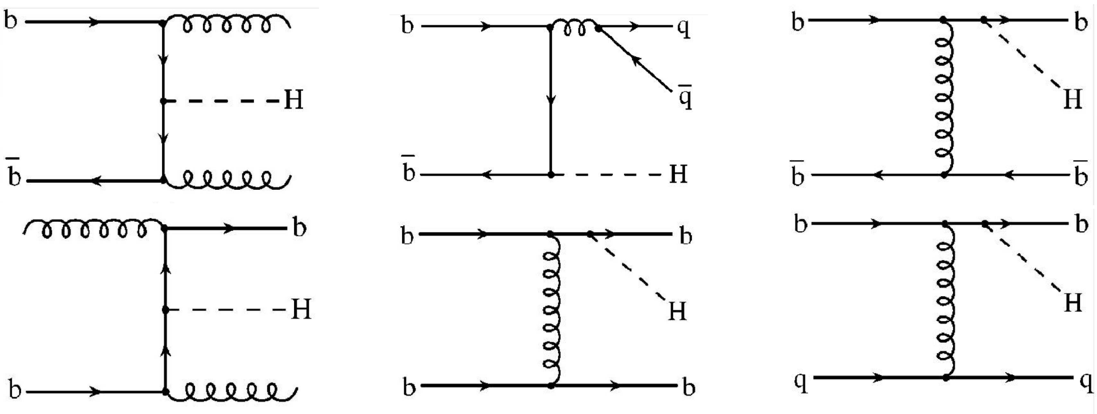

Figure 6: Next-to-next-to-leading contributions at 4FS; only representative diagrams are shown.

Therefore, in order to obtain all the contribution up to order α_s^2, in addition to the terms given in Eq. (92), we have to consider those coming from subprocesses starting at $O\left(\alpha_s^2\right)$ with initial-state heavy quarks, shown in Fig. (6). Therefore we have the new $O\left(\alpha_s^2\right)$ correction term:

$$\Delta\bar{\sigma} = \int dx_1 dx_2 \, f_b^{(4)}\left(x_1\right) f_b^{(4)}\left(x_2\right) \hat{\sigma}^{(4)}_{bb}\left(x_1, x_2, \frac{Q^2}{m_b^2}, \alpha_s^{(4)}\left(Q^2\right)\right)$$
$$+ \int dx_1 dx_2 \, \left[f_b^{(4)}\left(x_1\right) f_q^{(4)}\left(x_2, Q^2\right) + \left(x_1 \to x_2\right)\right] \hat{\sigma}^{(4)}_{bq}\left(x_1, x_2, \frac{Q^2}{m_b^2}, \alpha_s^{(4)}\left(Q^2\right)\right), \quad (128)$$

and, since all these subprocesses start at order α_s^2, we can replace $f^{(4)}$ and $\alpha_s^{(4)}$ with $f^{(5)}$ and $\alpha_s^{(5)}$. Therefore the new correction terms deriving from these new subprocesses have the form

$$\Delta\bar{\sigma} = \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 \int dx_1 dx_2 \, f_b^{(5)}\left(x_1\right) f_b^{(5)}\left(x_2\right) \left[\hat{\sigma}^{(4),2}_{bb} - \hat{\sigma}^{(4,0),2}_{bb}\right]$$
$$+ \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 \int dx_1 dx_2 \, \left[f_b^{(5)}\left(x_1\right) f_q^{(5)}\left(x_2, Q^2\right) + \left(x_1 \to x_2\right)\right] \left[\hat{\sigma}^{(4),2}_{bq} - \hat{\sigma}^{(4,0),2}_{bq}\right]. \quad (129)$$

As for the terms of Eq. (92), they now have to be included up to order α_s^2. Therefore we need to farther extend the matching condition Eq. (95), (96) and

(101) up to the second order, finding new contributions: from the change in running coupling, from the second iteration of the leading-order splitting functions $P_{ij}^{(0)}$ and from the next-to-leading-order splitting functions $P_{ij}^{(1)}$. The correction terms previously computed become (for simplicity we omit the terms obtained by exchanging $x_1 \leftrightarrow x_2$, and the dependence from x_1, x_2 is understood)

$$\begin{aligned}
\Delta\sigma_1 = &\int dx_1 dx_2 \\
&\left\{f_b^{(5)}\left(Q^2\right) + \alpha_s^{(5)}\left(Q^2\right) C_{b,1} + \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 C_{b,2}\right\} \\
&\left\{f_{\bar{b}}^{(5)}\left(Q^2\right) + \alpha_s^{(5)}\left(Q^2\right) C_{\bar{b},1} + \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 C_{\bar{b},2}\right\} \\
&\left[\left(\hat{\sigma}_{b\bar{b}}^{(4),0} - \hat{\sigma}_{b\bar{b}}^{(4,0),0}\right) + \left(\hat{\sigma}_{b\bar{b}}^{(4),1} - \hat{\sigma}_{b\bar{b}}^{(4,0),1}\right) \alpha_s^{(4)}\left(Q^2\right) + \left(\hat{\sigma}_{b\bar{b}}^{(4),2} - \hat{\sigma}_{b\bar{b}}^{(4,0),2}\right)\left(\alpha_s^{(4)}\left(Q^2\right)\right)^2\right],
\end{aligned} \tag{130}$$

from which we obtain, accounting also for the coupling constant through Eq. (17):

$$\begin{aligned}
&f_b^{(5)}\left(Q^2\right) f_{\bar{b}}^{(5)}\left(Q^2\right)\left(\hat{\sigma}_{b\bar{b}}^{(4),0} - \hat{\sigma}_{b\bar{b}}^{(4.0),0}\right) \\
&+\left\{f_b^{(5)}\left(Q^2\right) f_{\bar{b}}^{(5)}\left(Q^2\right)\left(\hat{\sigma}_{b\bar{b}}^{(4),1} - \hat{\sigma}_{b\bar{b}}^{(4,0),1}\right) + f_b^{(5)}\left(Q^2\right) C_{\bar{b},1}\left(\hat{\sigma}_{b\bar{b}}^{(4),0} - \hat{\sigma}_{b\bar{b}}^{(4,0),0}\right)\right. \\
&\qquad\left. + f_{\bar{b}}^{(5)}\left(Q^2\right) C_{b,1}\left(\hat{\sigma}_{b\bar{b}}^{(4),0} - \hat{\sigma}_{b\bar{b}}^{(4,0),0}\right)\right\}\alpha_s^{(5)}\left(Q^2\right) \\
&+\left\{f_b^{(5)}\left(Q^2\right) f_{\bar{b}}^{(5)}\left(Q^2\right)\left[\left(\hat{\sigma}_{b\bar{b}}^{(4),2} - \hat{\sigma}_{b\bar{b}}^{(4,0),2}\right) - \frac{2T_R}{3\left(2\pi\right)} L\left(\hat{\sigma}_{b\bar{b}}^{(4),1} - \hat{\sigma}_{b\bar{b}}^{(4,0),1}\right)\right]\right. \\
&\qquad + f_b^{(5)}\left(Q^2\right) C_{\bar{b},2}\left(\hat{\sigma}_{b\bar{b}}^{(4),0} - \hat{\sigma}_{b\bar{b}}^{(4,0),0}\right) + f_{\bar{b}}^{(5)}\left(Q^2\right) C_{b,2}\left(\hat{\sigma}_{b\bar{b}}^{(4),0} - \hat{\sigma}_{b\bar{b}}^{(4,0),0}\right) \\
&\qquad + f_b^{(5)}\left(Q^2\right) C_{\bar{b},1}\left(\hat{\sigma}_{b\bar{b}}^{(4),1} - \hat{\sigma}_{b\bar{b}}^{(4,0),1}\right) + f_{\bar{b}}^{(5)}\left(Q^2\right) C_{b,1}\left(\hat{\sigma}_{b\bar{b}}^{(4),1} - \hat{\sigma}_{b\bar{b}}^{(4,0),1}\right) \\
&\qquad\left. + C_{b,1} C_{\bar{b},1}\left(\hat{\sigma}_{b\bar{b}}^{(4),0} - \hat{\sigma}_{b\bar{b}}^{(4,0),0}\right)\right\}\left(\alpha_s^{(5)}\left(Q^2\right)\right)^2.
\end{aligned} \tag{131}$$

In order to find the coefficient $C_{b,1}$ and $C_{b,2}$ we expand Eq. (95) to one more order:

$$\begin{aligned}
f_b^{(4)} =& f_b^{(5)}\left(Q^2\right) - \alpha_s^{(4)}\left(Q^2\right) K_{bb}^{(1)} \otimes f_b^{(4)} - \left(\alpha_s^{(4)}\left(Q^2\right)\right)^2 K_{bb}^{(2)} \otimes f_b^{(4)} \\
&- \alpha_s^{(5)}\left(Q^2\right) P_{qq}^{(5),0} L \otimes f_b^{(5)}\left(Q^2\right) \\
&- \alpha_s^{(5)}\left(Q^2\right) P_{qg}^{(5),0} L \otimes f_g^{(5)}\left(Q^2\right) \\
&- \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 \sum_{i=q,\bar{q},g,b,\bar{b}} P_{bi}^{(5),1} L \otimes f_i^{(5)}\left(Q^2\right)
\end{aligned} \tag{132}$$

and using Eqs. (17) and (95) we get

$$\begin{aligned}
C_{b,1} &= -\left[K_{bb}^{(1)} + LP_{qb}^{(5),0}\right] \otimes f_b^{(5)}\left(Q^2\right) - LP_{qg}^{(5),0} \otimes f_g^{(5)}\left(Q^2\right) \\
C_{b,2} &= \left[K_{bb}^{(1)} \otimes \left(K_{bb}^{(1)} + P_{qq}^{(5),0} L\right) + \frac{2T_R}{3\,(2\pi)} L K_{bb}^{(1)} \right.\\
&\qquad \left. - K_{bb}^{(2)} - LP_{bb}^{(5),1} - LP_{b\bar{b}}^{(5),1} - K_{b\bar{b}}^{(2)}\right] \otimes f_b^{(5)}\left(Q^2\right) \\
&\quad + \left[LK_{bb}^{(1)} \otimes P_{qg}^{(5),0} - LP_{bg}^{(5),1} - K_{bg}^{(2)}\right] \otimes f_g^{(5)}\left(Q^2\right) \\
&\quad - \left[\sum_{i=q,\bar{q}} LP_{bi}^{(5),1} + K_{bi}^{(2)}\right] \otimes f_i^{(5)}\left(Q^2\right).
\end{aligned} \tag{133}$$

In the same way replacing $b \to \bar{b}$ we obtain $C_{\bar{b},1}$ and $C_{\bar{b},2}$.

Concerning the correction terms $\Delta\sigma_2$ and $\Delta\sigma_3$ to order α_s^2, in order to find them we don't need to write the matching condition for the gluon at higher order than α_s as done in Eq. (101), since the partonic processes involving a gluon in the initial state start at order α_s. Therefore, from Eq. (104) we get

$$\begin{aligned}
\Delta\sigma_2 = \int dx_1 dx_2 \Big\{ & f_b^{(5)}\left(Q^2\right) f_g^{(5)}\left(Q^2\right) \left(\hat{\sigma}_{bg}^{(4),1} - \hat{\sigma}_{bg}^{(4,0),1}\right) \alpha_s^{(5)}\left(Q^2\right) \\
& + \left(f_b^{(5)}\left(Q^2\right) C_{g,1} + f_g^{(5)}\left(Q^2\right) C_{b,1}\right) \left(\hat{\sigma}_{bg}^{(4),1} - \hat{\sigma}_{bg}^{(4,0),1}\right) \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 \\
& + f_b^{(5)}\left(Q^2\right) f_g^{(5)}\left(Q^2\right) \left(\hat{\sigma}_{bg}^{(4),2} - \hat{\sigma}_{bg}^{(4,0),2}\right) \left(\alpha_s^{(5)}\left(Q^2\right)\right)^2 \Big\},
\end{aligned} \tag{134}$$

where

$$C_{g,1} = L \frac{2T_R}{3\,(2\pi)} f_g^{(5)}\left(Q^2\right) - K_{gb}^{(1)} \otimes f_b^{(5)}\left(x, Q^2\right). \tag{135}$$

Replacing $b \to \bar{b}$ we get $\Delta\sigma_3$.

In order to check if also at this order the difference term vanishes, we should derive the general expression of the massless limit of the 4FS in term of 5FS cross sections, as done at order α_s in the previous section. This is not done here explicitly. However the general lines to follow are those previously depicted, and starting from the result of this subsection, one may checked this also numerically.

5.4 Higher orders

The discussion provided so far shows how the results found by generalizing the FONLL method in the case of DIS are true also for the hadronic processes: namely the difference term vanishes and we find a total cross section obtained by a convolution between massless PDFs and massive partonic cross sections.

We may question if, as previously done in the DIS case, we can generalize the discussion to higher orders.

Once we account for the presence of initial state heavy quarks also in the 4FS, the general FONLL result would be

$$
\begin{aligned}
&\sigma^{FONLL} = \\
&\sum_{i,j=q,\bar{q},h,\bar{h},g} \; \sum_{l,m=q,\bar{q},h,\bar{h},g} \int dx_1 dx_2 \, K_{il}^{-1} \otimes f_l^{(5)}\left(Q^2\right) K_{jm}^{-1} \otimes f_m^{(5)}\left(Q^2\right) \\
&\left\{ \hat{\sigma}_{ij}^{(4)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(4)}\left(Q^2\right)\right) - \hat{\sigma}_{ij}^{(4,0)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(4)}\left(Q^2\right)\right)\right\} \\
&+ \sum_{i=q,\bar{q},h,\bar{h},g} \int dx_1 dx_2 \, f_i^{(5)}\left(x_1, Q^2\right) f_j^{(5)}\left(x_2, Q^2\right) \hat{\sigma}_{ij}^{(5)}\left(x_1, x_2, \alpha_s^{(5)}\left(Q^2\right)\right)
\end{aligned}
\tag{136}
$$

where we use the inverse of eq.(44) evolved at scale Q^2 to express $f^{(4)}$ in term of $f^{(5)}$. Requiring that the massless limit of the massive scheme total cross section is equal order by order in the same strong coupling to the massive scheme expression, we have that the second and third term in the equation above cancel order by order in perturbation theory, leaving

$$
\begin{aligned}
\sigma^{FONLL} = &\sum_{i,j=q,\bar{q},h,\bar{h},g} \; \sum_{l,m=q,\bar{q},h,\bar{h},g} \int dx_1 dx_2 \\
&K_{il}^{-1} \otimes f_l^{(5)}\left(Q^2\right) K_{jm}^{-1} \otimes f_m^{(5)}\left(Q^2\right) \left\{ \hat{\sigma}_{ij}^{(4)}\left(x_1, x_2, \frac{Q^2}{m^2}, \alpha_s^{(4)}\left(Q^2\right)\right)\right\}.
\end{aligned}
\tag{137}
$$

Thus we find that the difference term, which has been explicitly verified to vanish at order α_s^0 and α_s in the previous subsection, vanishes identically also at higher orders.

6 Massive partonic cross section

In this section we provide results about the computation of the hadronic cross sections $\hat{\sigma}^{(4)}_{b\bar{b}}$ and $\hat{\sigma}^{(4)}_{bg}$, which are required in order to obtain a full analytic expression of Eqs. (115) and (116). This computation, in addition to completing the analytical expression of the problem, allows one to see the specific features of the 4FS, finding out the collinear divergences coming from the emissions of heavy quarks which explicitly appears in the massive results. Depending on the kind of regularization we choose to deal with these singularities, partonic cross sections can be calculated in two different ways: we can choose whether to use dimensional regularization or to introduce an explicit heavy quark mass which acts as an infrared regulator, leading to the large logarithms of Q^2/m_b^2. After defining the kinematics in the proper way, we will first present results concerning calculations using dimensional regularization, reporting results obtained in Ref. [9], and using them in order to obtain the final analytical expression of Eqs. (115) and (116). We will then perform full computations of the partonic cross sections using the heavy quark mass to regularized the collinear divergences, pointing out the main features of the calculation step by step and highlighting its main features. This is interesting because of different reasons: first, despite in Ref. [9] explicit results are reported, this computation was done only once many years ago, therefore, for sure, it is useful to check it. Secondly, in Ref. [9] the results are shown without any intermediate calculations, which are required in order to fully understand the main feature of the addressed processes. Finally, computations performed using dimensional regularization, despite giving very simple analytical expressions , do not allow one to see explicitly what is happening, namely the large collinear logarithms arising thanks to heavy quark emissions do not appear explicitly. Because of these motivations it is useful to fully check this computation using an explicit heavy quark mass to regularize the collinear divergences. These results should be checked numerically, to ensure they are equivalent to those obtained by dimensional regularization of Ref. [9].

6.1 Kinematics

In this subsection we define the proper kinematics of the problem, in such a way that it works for both massless and massive particles as well. In the standard deep inelastic scattering the cinematic of the problem is defined by an invariant quantity given by a suitable combination of dimensional parameters of the problem: the Bjorken variable

$$x = \frac{Q^2}{2p \cdot q}, \tag{138}$$

where p is the 4-momentum of the hadron. We then introduce the corresponding partonic variable $\hat{x}$ replacing the hadron momentum p with the partonic one $\hat{p}$ and we define the variable z as

$$\hat{x} = zx. \tag{139}$$

Giving these definitions, perturbative factorization applies with respect to the variable z. These definitions are usually given in the massless case but they still work with massive partons.

We would like to do the same in a generic hadronic process, considering at a parton level two massive quarks in the initial state. We have to define in a covariant way two variables x_1, x_2 with respect to which perturbative factorization applies. Thus we will be able to write the hadronic cinematic in terms of the partonic one through the standard convolution

$$\begin{aligned} &\int dx_1 dx_2\, f_1(x_1)\, f_2(x_2)\, \hat{\sigma}(x_1, x_2, \hat{\tau}) = \int \frac{dy}{y} L(y)\, \hat{\sigma}\left(\frac{\hat{\tau}}{y}\right) \\ &L(y) = \int \frac{dz}{z} f_1\left(\frac{y}{z}\right) f_2(z)\,. \end{aligned} \tag{140}$$

We start defining the invariant quantity for our process

$$\tau = \frac{m_H^2}{s}, \tag{141}$$

which in the partonic system is given by

$$\hat{\tau} = \frac{m_H^2}{\hat{s}}. \tag{142}$$

We define the product $x_1 x_2$ with

$$\hat{s} = x_1 x_2 s. \tag{143}$$

In order to define the single variables x_1, and x_2 we have to specify the boost which brings from the hadronic centre-of-mass frame of reference to the partonic one. The collision is collinear, so there exists a partonic centre of mass frame in which $\hat{p}_A + \hat{p}_B = 0$, which is a boost of the hadronic one. So we define

$$\begin{aligned} x_1 &= \sqrt{\tau} e^{+y} \\ x_2 &= \sqrt{\tau} e^{-y} \end{aligned} \tag{144}$$

where y is the rapidity of the system given by

$$y = \frac{1}{2} \log \frac{1+\beta}{1-\beta} \tag{145}$$

and β defines the boost from the hadronic centre of mass frame of reference to the partonic one. Dealing with massless particles the rapidity coincides with the pseudorapidity, and we find the standard definitions of the kinematics of hadronic collisions involving massless partons. For a process with energy in the centre of mass frame given by $\sqrt{\tau s}$ the rapidity is limited by $\mid y \mid < -\frac{1}{2}\log\tau$, so that x_1, x_2 are contained in $(0, 1)$.

We will now work out explicit expressions for the energy and the momentum of the partons entering the hard collision, in terms of the scaling variables x_1 and x_2. Let us consider two partons A, B in a collinear collision. In the partonic centre of mass frame we have

$$\begin{aligned} \hat{p}_A &= \left(\hat{E}_A, 0, 0, \hat{p}_A\right) \\ \hat{p}_B &= \left(\hat{E}_B, 0, 0, \hat{p}_B\right) = \left(\hat{E}_A, 0, 0, -\hat{p}_A\right), \end{aligned} \tag{146}$$

with

$$\hat{E}_i = \sqrt{\mid p_i \mid + m_i^2}. \tag{147}$$

Using the definition of the product $x_1 x_2$ we find

$$\hat{s} = (\hat{p}_A + \hat{p}_B)^2 = \left(\hat{E}_A + \hat{E}_B\right)^2 = x_1 x_2 (p_A + p_B)^2 = 4 x_1 x_2 E_P^2. \tag{148}$$

Substituiting eq.(147) in eq.(148) and solving respect to $\mid p_A \mid$ we find

$$\mid p_A \mid^2 = \frac{8 (x_1 x_2)^2 E_P^4 - 8 x_1 x_2 m_A^2 - 8 x_1 x_2 m_B^2 + (m_A^2 - m_B^2)^2}{16 x_1 x_2 E_P^2}. \tag{149}$$

Thus the partonic quantities expressed in terms of the variable x_1, x_2 and of the colliding protons energy are

$$\begin{aligned} \hat{E}_i &= \sqrt{\mid \hat{p}_i \mid^2 + m_i^2} \qquad\qquad i = A, B \\ \mid \hat{p}_A \mid &= \sqrt{\frac{8 (x_1 x_2)^2 E_P^4 - 8 x_1 x_2 m_A^2 - 8 x_1 x_2 m_B^2 + (m_A^2 - m_B^2)^2}{16 x_1 x_2 E_P^2}}. \end{aligned} \tag{150}$$

In order to obtain these quantities in the hadronic frame we apply the boost β

$$\begin{aligned} \hat{E}_A{}' &= \gamma\left(\hat{E}_A + \beta \hat{p}_A\right), & \hat{E}_B{}' &= \gamma\left(\hat{E}_B - \beta \hat{p}_B\right), \\ \hat{p}_A{}' &= \gamma\left(\hat{p}_A + \beta \hat{E}_A\right), & \hat{p}_B{}' &= \gamma\left(-\hat{p}_B + \beta \hat{E}_B\right), \end{aligned} \tag{151}$$

with

$$\beta = \frac{x_1 - x_2}{x_1 + x_2}. \tag{152}$$

In our example of hadronic process we are interested in the hard collisions between a bottom quark and its antiparticle or between a bottom quark/antiquark and a gluon. In these particular cases we have

$$m_A = m_B = m_b,$$

and

$$m_A = m_b, \;\; m_B = 0,$$

respectively, and Eq. (150) becomes in the first case

$$\begin{aligned} \hat{E}_A = \hat{E}_B &= \sqrt{x_1 x_2} E_P \\ \mid \hat{p}_A \mid &= \sqrt{x_1 x_2 E_P^2 - m_b^2}, \end{aligned} \tag{153}$$

with

$$\hat{s} > (2m_b)^2 \rightarrow \quad x_1 x_2 E_P^2 - m_b^2 > 0, \tag{154}$$

while in the second one

$$\begin{aligned} \hat{E}_A &= \frac{4x_1 x_2 E_P^2 + m_b^2}{4\sqrt{x_1 x_2} E_p} \\ \hat{E}_B &= \mid \hat{p}_A \mid \\ \mid \hat{p}_A \mid &= \frac{4x_1 x_2 E_P^2 - m_b^2}{4\sqrt{x_1 x_2 E_P^2}}, \end{aligned} \tag{155}$$

with

$$\hat{s} > (m_b)^2 \rightarrow \quad 4x_1 x_2 E_P^2 - m_b^2 > 0. \tag{156}$$

To sum up, in this subsection we have introduced general definitions for the kinematics of the problem, taking into account the presence of massive partons in the initial state, defining the variable x_1 and x_2 which allow us to write the hadronic cinematic in terms of the partonic one. Then we have worked out the explicit expressions of the partonic quantities in terms of x_1 and x_2.

6.2 Dimensional regularization

In this subsection we report the results of Ref. [9] regarding the partonic cross sections $\hat{\sigma}^{(4)}_{b\bar{b}}$ and $\hat{\sigma}^{(4)}_{bg}$ at NLO, computed using dimensional regularization. Eventually, using these results, we will give our final expression of Eqs. (115) and (116).

To NLO the 4FS partonic cross section has to be computed up to order α_s, therefore the total cross section receives contributions from the following subprocesses:

- $O(1) \Rightarrow b\bar{b} \rightarrow H$
- $O(\alpha_s) \Rightarrow b\bar{b} \rightarrow H$(1-loop), $bg \rightarrow Hb$, $\bar{b}g \rightarrow H\bar{b}$, $b\bar{b} \rightarrow Hg$

The spin and color averaged cross section for the leading order subprocess $b\bar{b} \rightarrow H$, shown in the first diagram of Fig. (7) is

$$\hat{\sigma}^{(4),0}_{b\bar{b}} = \frac{\pi}{6} \frac{\bar{m}^2(\mu)}{v^2} \mu^{2\epsilon} \frac{1}{m_H^2} \delta(1-z), \tag{157}$$

where we have defined

$$z \equiv \hat{\tau} = \frac{m_H^2}{\hat{s}}. \tag{158}$$

The calculation is performed in $N = 4 - 2\epsilon$ dimensions, μ is the so-called Hooft mass, introduced so that the renormalized Yukawa coupling is dimensionless in N dimension.

After calculating the leading order term, we have to consider the $O(\alpha_s)$ interference of the 1-loop vertex correction with the three diagram of Fig. (7). We find

$$\begin{aligned} \hat{\sigma}^{(4),1}_{b\bar{b}\rightarrow H} = & \frac{\pi}{6} \frac{\bar{m}^2(\mu)}{v^2} \mu^{2\epsilon} \frac{1}{\hat{s}} \delta(1-z) \left[C_F \frac{\alpha_s}{2\pi} \mu^{2\epsilon} \left(\frac{4\pi}{m_H^2} \right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \right. \\ & \left. \times \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 2 + \frac{2\pi^2}{3} \right) \right], \end{aligned} \tag{159}$$

where $C_F = 4/3$.

In writing the expressions above we have to take into account an additional feature which does not appear in other similar calculations, such as the Drell-Yan process: the ultraviolet renormalization of the Yukawa coupling. In fact the heavy quark mass has to be renormalized, and we denote by $\bar{m}(\mu)$ the renormalized heavy quark mass obtained subtracting the ultraviolet divergence with the $\bar{MS}$ scheme. Therefore we have new counterterm

$$-\mu^{\epsilon} \frac{\bar{m}(\mu)}{v} \left(1 - \frac{\delta \bar{m}}{\bar{m}} \right) b\bar{b}H \tag{160}$$

where

$$\frac{\delta \bar{m}}{\bar{m}} = C_F \frac{\alpha_s}{4\pi} 3 \left(\frac{1}{\epsilon} - \gamma + \log 4\pi \right). \tag{161}$$

We notice this cross section displays both a collinear and an infrared divergence, given respectively by the $\left(\frac{1}{\epsilon}\right)$ and $\left(\frac{1}{\epsilon^2}\right)$ terms. The divergence that produces the $\left(\frac{1}{\epsilon}\right)$ term concerns heavy quarks collinear emissions, which, if heavy quark mass was used as regulator would appear as the large logarithm which are resummed to all orders in the 5FS but which explicitly appears in the 4FS.

Furthermore, the IR divergence $\left(\frac{1}{\epsilon^2}\right)$ will disappear after adding the contribution coming from real emissions.

As for the emission of a real gluon, for which Feynman diagrams are shown in Fig. (10), the regularized cross section is found to be

$$\begin{aligned}\hat{\sigma}^{(4),1}_{b\bar{b}\to Hg} = & C_F \frac{\alpha_s}{12} \mu^{2\epsilon} \frac{\bar{m}^2(\mu)}{v^2} \frac{1}{\hat{s}} \left(\frac{4\pi}{m_H^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \\ & \left[\frac{2}{\epsilon^2}\delta(1-z) - \frac{2}{\epsilon}\frac{1+z^2}{(1-z)_+} - 2\left(1+z^2\right)\frac{\log z}{1-z} \right. \\ & \left. + 4\left(1+z^2\right)\left(\frac{\log(1-z)}{1-z}\right)_+ + 2(1-z)\right],\end{aligned} \tag{162}$$

with the plus prescription defined as

$$\int_0^1 dz \left[f(z)\right]_+ h(z) = \int_0^1 dz f(z)\left[h(z) - h(1)\right]. \tag{163}$$

Again, we notice how the result contains both collinear and IR divergences. In order to obtain the full expression of Eq. (116), we now have to add up the cross sections for real and virtual emissions and, as seen in Eq. (116), the terms referred to the collinear divergences, which explicitly appear in the 4FS cross sections, have to be removed. In order to do this we have to consider the analogue of Eq. (118) in dimensional regularization, which is

$$\begin{aligned}f_b\left(x,Q^2\right) = & -\alpha_s\left(Q^2\right)\int_x^1 \frac{dy}{y} P_{qg}\left(\frac{x}{y}\right) f_g\left(y,Q^2\right)\left(\frac{1}{\epsilon} - \gamma + \log 4\pi\right) \\ & - \alpha_s\left(Q^2\right)\int_x^1 \frac{dy}{y} P_{qq}\left(\frac{x}{y}\right) f_b\left(y,Q^2\right)\left(\frac{1}{\epsilon} - \gamma + \log 4\pi\right),\end{aligned} \tag{164}$$

where we see how the collinear divergences appear as a factor $\left(\frac{1}{\epsilon}\right)$ rather than as a logarithm $L = \log\frac{Q^2}{m_b^2}$. Therefore, as for the process $b\bar{b} \to H$, summing the corrections due to real and virtual emissions and removing the collinear divergences, whose contribution is written as $\hat{\sigma}_{cd}$ in the equation below , we get

$$\begin{aligned}\hat{\hat{\sigma}}^{(4),1}_{b\bar{b}} = & \hat{\sigma}^{(4),1}_{b\bar{b}\to Hg} + \hat{\sigma}^{(4),1}_{b\bar{b}\to H} - \hat{\sigma}_{cd} = \frac{\pi}{6}\frac{\bar{m}^2(\mu)}{v^2}\frac{\alpha_s}{\hat{s}} \\ & \times \left\{\frac{1}{\pi}P_{qq}(z)\log\frac{m_H^2}{Q^2} \right. \\ & + C_F\frac{1}{2\pi}\left[\left(-2 + \frac{2\pi^2}{3} - 3\log\frac{m_H^2}{Q^2}\right)\delta(1-z) \right. \\ & \left.\left. - 2\left(1+z^2\right)\frac{\log z}{1-z} + 4\left(1+z^2\right)\left(\frac{\log(1-z)}{1-z}\right)_+ + 2(1-z)\right]\right\}.\end{aligned} \tag{165}$$

This is the sum of Eq. (162) and (159), where the term proportional to $(1/\epsilon - \gamma + \log 4\pi)$ has been remove by renormalization, and where the limit $\epsilon \to 0$ has been taken. We notice that the IR divergence cancels between real and virtual emissions, leaving a result which is finite both in the UV and IR limits. Since Eq. (165) is obtained removing both IR and UV divergences, it depends on two different scale. The first one, μ, is that introduced by UV renormalization of the heavy quark mass which appears in the Yukawa coupling. In fact after renormalization the heavy quark mass satisfies a running equation of the form

$$\bar{m}(\mu) = \bar{m}(\mu_0)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{1}{\beta_0}},$$
$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1+\beta_0\left[\alpha_s(\mu_0)/\pi\right]\log\mu^2/\mu_0^2}, \tag{166}$$

which at NLO is

$$\bar{m}(\mu) = \bar{m}(\mu_0)\left[1 - C_F\frac{\alpha_s}{4\pi}3\log\frac{\mu^2}{\mu_0^2}\right], \tag{167}$$

and therefore it depends on a renormalization scale μ. As for the second scale which appears in Eq. (165), Q^2, it is the one referred to the factorization, associated to the parton distribution functions.

We now consider the subprocesses $bg \to bH$ and $\bar{b}g \to \bar{b}H$, whose cross sections are equal and whose Feynman diagrams are shown in Fig. (11). Using result of Ref. [9], the spin and color average cross section is

$$\hat{\sigma}_{gb}^{(4),1} = \frac{\alpha_s}{12}\frac{\bar{m}^2(\mu)}{v^2}\mu^{4\epsilon}\frac{1}{\hat{s}}\left\{P_{qg}(z)\left[-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \log\left(\frac{m_H^2}{4\pi}\frac{(1-z)^2}{z}\right)\right] + \frac{1}{4}(1-z)(7z-3)\right\}, \tag{168}$$

where the collinear divergence due to the heavy quark emission is the term proportional to $\left(\frac{1}{\epsilon}\right)$. Following Eq. (116) we subtract the collinear divergence, obtaining

$$\hat{\hat{\sigma}}_{gb}^{(4),1} = \frac{\alpha_s}{12}\frac{\bar{m}^2(\mu^2)}{v^2}\frac{1}{\hat{s}}\left[P_{qg}(z)\log\left(\frac{m_H^2}{Q^2}\frac{(1-z)^2}{z}\right) + \frac{1}{4}(1-z)(7z-3)\right] \tag{169}$$

To sum up, we have presented explicit result for the massive cross sections which appears in Eqs. (115) and (116), obtained using dimensional regularization to treat the collinear divergences due to heavy quark emissions. The final analytic result up to order $O(\alpha_s)$ for the total cross section of Higgs production from bottom quarks fusion in the generalized FONLL scheme is therefore given by Eqs. (115), (119), (157), (165) and (169) which are reported below:

$$\sigma^{FONLL-A,(0)} = \int dx_1 dx_2 \left[f_{\bar{b}}^{(5)}\left(x_1, Q^2\right) f_{b}^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right) \right] \hat{\sigma}_{b\bar{b}}^{(4),0}\left(x_1, x_2\right), \tag{170}$$

with

$$\hat{\sigma}_{b\bar{b}}^{(4),0} = \frac{\pi}{6} \frac{\bar{m}^2(\mu)}{v^2} \mu^{2\epsilon} \frac{1}{m_H^2} \delta(1-z), \tag{171}$$

$$\begin{aligned} \sigma^{FONLL-A,(1)} &= \int dx_1 dx_2 \\ &\left[f_{\bar{b}}^{(5)}\left(x_1, Q^2\right) f_{b}^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right) \right] \hat{\hat{\sigma}}_{b\bar{b}}^{(4),1} \\ &+ \left[f_{b}^{(5)}\left(x_1, Q^2\right) f_{g}^{(5)}\left(x_2, Q^2\right) + \left(x_1 \leftrightarrow x_2\right) + \left(b \to \bar{b}\right) \right] \hat{\hat{\sigma}}_{bg}^{(4),1} \\ &+ \left\{ - \left(K_{bb}^{(1)} \otimes f_{\bar{b}}^{(5)}\left(x_2, Q^2\right) \right) f_{b}^{(5)}\left(x_1, Q^2\right) + \left(b \leftrightarrow \bar{b}, x_1 \leftrightarrow x_2\right) + \left(x_1 \leftrightarrow x_2\right) \right\} \hat{\sigma}_{b\bar{b}}^{(4),0}, \end{aligned} \tag{172}$$

with

$$\begin{aligned} \hat{\hat{\sigma}}_{b\bar{b}}^{(4),1} &= \frac{\pi}{6} \frac{\bar{m}^2(\mu)}{v^2} \frac{1}{\hat{s}} \times \left\{ \delta(1-z) + \frac{\alpha_s}{\pi} P_{qq}(z) \log \frac{m_H^2}{Q^2} \right. \\ &+ C_F \frac{\alpha_s}{2\pi} \left[\left(-2 + \frac{2\pi^2}{3} - 3 \log \frac{m_H^2}{Q^2} \right) \delta(1-z) \right. \\ &\left. \left. - 2\left(1+z^2\right) \frac{\log z}{1-z} + 4\left(1+z^2\right) \left(\frac{\log(1-z)}{1-z} \right)_+ + 2(1-z) \right] \right\}, \end{aligned} \tag{173}$$

and

$$\hat{\hat{\sigma}}_{gb}^{(4),1} = \frac{\alpha_s}{12} \frac{\bar{m}^2(\mu^2)}{v^2} \frac{1}{\hat{s}} \left[P_{qg}(z) \log \left(\frac{m_H^2}{Q^2} \frac{(1-z)^2}{z} \right) + \frac{1}{4}(1-z)(7z-3) \right]. \tag{174}$$

6.3 Massive regularization

In the previous section we worked out the full analytic results for the FONLL method with initial-state heavy quarks, using the expression for partonic cross section at NLO calculated in dimensional regularization in Ref. [9]. The same computation of the partonic cross sections may be performed using an explicit heavy quark mass as IR regulator to treat the collinear divergence coming from the emission of a massive parton. The aim of this section is to perform full calculation of massive partonic cross sections up to NLO, using the heavy quark mass to regularized the collinear singularities. This allows us to check the results of Ref. [9], using a different kind of regularization. Despite the results used in the previous subsection are simpler and easier to use, we use this part to highlight the main features of the computations, following each step, and finding explicitly the features of the massive scheme computations. The results should be checked numerically, to ensure they are equivalent to those of the previous subsection.

6.4 The leading order subprocess $b\bar{b} \to H$

We now work out the LO contribution to the process $b\bar{b} \to H$, whose Feynman diagram is the first shown in Fig. (7).

Figure 7: Leading-order and next-to-leading-order contributions to the four flavour scheme.

From standard Feynman QCD rules, the amplitude for the process shown in Fig. (7) is given by

$$iM = \bar{v}^{s_2}(p')\left(-i\frac{m_b}{v}\right)u^{s_1}(p) \tag{175}$$

We take the square amplitude and average over the initial polarizations and colors of the two quarks

$$\begin{aligned}\frac{1}{2^2}\frac{1}{3^2}\sum_{s_1,s_2}\sum_{colors} \mid M \mid^2 &= \frac{1}{12}\frac{m_b^2}{v^2}\sum_{s_1,s_2}\bar{v}^{s_2}(p')\,u^{s_1}(p)\,\bar{u}^{s_1}(p)\,v^{s_2}(p') \\ &= \frac{1}{12}Tr\left[(\not{p}+m_b)(\not{p}'-m_b)\right] \\ &= \frac{1}{3}\frac{m_b^2}{v^2}\left[(p\cdot p')-m_b^2\right].\end{aligned} \tag{176}$$

Adding the flux factor and the phase space we get

$$\sigma_0 \equiv \hat{\sigma}^{(4),0}_{b\bar{b}} = \frac{\frac{1}{2^2}\frac{1}{3^2}\sum_{s_1,s_2}\sum_{colors} \mid M \mid^2}{4\sqrt{\left(p\cdot p'\right)^2 - m_b^4}}\, d\phi_1$$

$$\begin{aligned} &= \frac{m_b^2}{v^2}\frac{\left[\left(p\cdot p'\right) - m_b^2\right]}{12\sqrt{\left(p\cdot p'\right)^2 - m_b^4}}\frac{d^3 p_f}{\left(2\pi\right)^3 2E_f} \\ &\quad \left(2\pi\right)^4 \delta^{(4)}\left(p + p' - p_f\right)\, dp_f^0\, \delta\left(p_f^2 - m_H^2\right) \end{aligned} \tag{177}$$

$$= \frac{m_b^2}{v^2}\frac{\pi\left[\left(p\cdot p'\right) - m_b^2\right]}{6\sqrt{\left(p\cdot p'\right)^2 - m_b^4}}\frac{1}{2E_f}\frac{1}{m_H^2}\delta\left(1-\hat{\tau}\right)$$

$$= \frac{m_b^2}{v^2}\frac{\pi}{6}\left(1 - \frac{4m_b^2}{m_H^2}\right)^{\frac{1}{2}}\frac{1}{2\left(E+E'\right)}\frac{1}{m_H^2}\delta\left(1-\hat{\tau}\right).$$

6.5 The next-to-leading order subprocess $b\bar{b} \to H$(1-loop)

We proceed to calculate the virtual correction of order α_s to the subprocess $b\bar{b} \to H$, corresponding to the second Feynman diagram in Fig. (7). We define

$$\begin{aligned} m &\equiv m_b \\ k' &= q - k \end{aligned} \tag{178}$$

Using Feynamn rules we obtain the amplitude

$$\begin{aligned} iA_1 = &\int \frac{d^4k}{\left(2\pi\right)^4}\frac{-ig_{\nu\rho}}{\left(k-p\right)^2 + i\epsilon}\bar{v}\left(p'\right)\left(-igt^A\gamma^\nu\right)\frac{i\left(-\not{k}' + m\right)}{k'^2 - m^2 + i\epsilon} \\ &\quad \left(-i\frac{m}{v}\right)\frac{i\left(\not{k} + m\right)}{k^2 - m^2 + i\epsilon}\left(-igt^A\gamma^\rho\right)u\left(p\right) \\ &= \left(-g^2\right)\left(t^A t^A\right)\frac{m}{v}\int\frac{d^4k}{\left(2\pi\right)^4}\frac{\bar{v}\left(p'\right)\gamma^\nu\left(-\not{k}' + m\right)\left(\not{k} + m\right)\gamma_\nu u\left(p\right)}{\left(\left(k-p\right)^2 + i\epsilon\right)\left(k'^2 - m^2 + i\epsilon\right)\left(k^2 - m^2 + i\epsilon\right)}. \end{aligned} \tag{179}$$

In order to simplify this expression we introduce three Feynman parameters x, y and z

$$\frac{1}{\left((k-p)^2+i\epsilon\right)\left(k'^2-m^2+i\epsilon\right)\left(k^2-m^2+i\epsilon\right)}$$

$$= \int_0^1 dxdydz \frac{2\delta(x+y+z-1)}{\left[x\left(k^2-m^2\right)+y\left(k'^2-m\right)+z(k-p)^2+i\epsilon\right]^3} \tag{180}$$

$$= \int_0^1 dxdydz \frac{2\delta(x+y+z-1)}{\left[k^2-2k\cdot(yq+zp)+yq^2+zp^2-(x+y)m^2+i\epsilon\right]^3}.$$

We define the variable

$$l = k - yq - zp, \tag{181}$$

and we rewrite numerator and denominator in terms of l. For the denominator we have

$$\begin{aligned}
&k^2-2k\cdot(yq+zp)+yq^2+zp^2-(x+y)m^2+i\epsilon \\
&= (k-yq-zp)^2-y^2q^2-z^2p^2-2yz\,q\cdot p+yq^2+zp^2-(1-z)m^2+i\epsilon \\
&= l^2+yq^2(1-y)-m^2(1-z)^2-2yz\,q\cdot p+i\epsilon \\
&= l^2-m^2(1-z)^2+y\left[q^2(1-y)-2z\,p\cdot q\right] \\
&= l^2-m^2(1-z)^2+xyq^2+i\epsilon \\
\\
&= l^2-\Delta+i\epsilon \equiv D,
\end{aligned} \tag{182}$$

with

$$\Delta = -xyq^2+m^2(1-z), \tag{183}$$

and where we have used the identity $q^2(1-y)-2z\,p\cdot q = xq^2$ based on

$$q^2-2p\cdot q=0$$

.

For the nominator we find

$$\begin{aligned}
&\bar{v}(p')\gamma^{\nu}(-\not{k}'+m)(\not{k}+m)\gamma_{\nu}u(p) \\
&= \bar{v}(p')\left[-4(k'\cdot k)+2m(\not{k}'-\not{k})+4m^2\right]u(p) \\
&= \bar{v}(p')[4l^2-4y(1-y)q^2-4z(1-2y)(q\cdot p)+4m^2z^2 \\
&\qquad +2m(1-2y)(\not{p}+\not{p}')-4mz\,\not{p}+4m^2]u(p) \\
\\
&= \bar{v}(p')\left[4l^2-4y(1-y)q^2-2z(1-2y)q^2+4m^2z(z-1)+4m^2\right]u(p),
\end{aligned} \tag{184}$$

where we have used

$$\begin{aligned}
&\bar{v}\left(p'\right)\, \not{p}' = -m\bar{v}\left(p'\right) \\
&\not{p}u\left(p\right) = mu\left(p\right) \\
&q^2 - 2\,p\cdot q = 0,
\end{aligned} \tag{185}$$

and we have dropped odd terms of l. We finally get

$$\begin{aligned}
&iA_1 = iA_1^{UV} + iA_1^{IR} = \\
&\left(-g^2\right)\left(t^At^A\right)\frac{m}{v}\int_0^1 dxdydz\,\delta\left(x+y+z-1\right)\int\frac{d^4l}{\left(2\pi\right)^4}\frac{8l^2}{\left[l^2-\Delta+i\epsilon\right]^3}\,\bar{v}\left(p'\right)u\left(p\right) \\
&+\left(-g^2\right)\left(t^At^A\right)\frac{m}{v}\int_0^1 dxdydz\,\delta\left(x+y+z-1\right) \\
&2\int\frac{d^4l}{\left(2\pi\right)^4}\frac{\left[4m^2z\left(z-1\right)+4m^2-4y\left(y-1\right)q^2-2z\left(1-2y\right)q^2\right]}{\left[l^2-\Delta+i\epsilon\right]^3}\,\bar{v}\left(p'\right)u\left(p\right).
\end{aligned} \tag{186}$$

Therefore we note that the amplitude for the 1-loop correction to the process $b\bar{b}\to H$ is splitted into one part which is UV divergent, iA_1^{UV}, and a second part IR divergent, iA_1^{IR}.

6.6 Field strength renormalization

In the calculation of the previous section we did not take into account the effect of the field-strength renormalization, which is required in the calculation of NLO corrections. In fact we have to insert a factor $\sqrt{Z}$ for each quark, therefore, defining as Z_2 the field strength renormalization at order α_s and as A the total amplitude for the process $b\bar{b}\to H$, we have

$$Z_2A = \left(1+\delta Z_2\right)\left(A_0+A_1\right) = A_0+A_1+A_0\left(\delta Z_2\right). \tag{187}$$

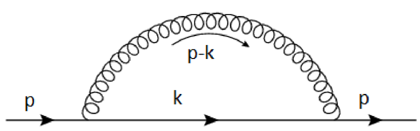

Figure 8: Feynman diagram for quark self energy at order α_s

In the following calculations m_0 will be the bare mass of the bottom quark while μ a fictitious mass of the gluon that we introduce in order to regularize the calculation. We start writing down the amplitude associated to the Feynman diagram in Fig. (8) which represents the quark self energy

$$
\begin{aligned}
-i\Sigma_2(p) &= \int \frac{d^4k}{(2\pi)^4}\left(-igt^A\gamma^\mu\right)\frac{i\left(\not{k}+m_0\right)}{k^2-m_0^2+i\epsilon}\left(-igt^A\gamma^\nu\right)\frac{-ig_{\mu\nu}}{(p-k)^2-\mu^2+i\epsilon} \\
&= \left(-g^2\right)\left(t^At^A\right)\int \frac{d^4k}{(2\pi)^4}\frac{\gamma^\mu\left(\not{k}+m_0\right)\gamma_\mu}{\left(k^2-m_0^2+i\epsilon\right)\left((p-k)^2-\mu^2+i\epsilon\right)} \\
&= \left(-g^2\right)\left(t^At^A\right)\int_0^1 dx \int \frac{d^4l}{(2\pi)^4}\frac{-2x\,\not{p}+4m_0}{\left[l^2-\Delta+i\epsilon\right]^2},
\end{aligned} \tag{188}
$$

with

$$
\begin{aligned}
l &= k - xp \\
\Delta &= -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2,
\end{aligned} \tag{189}
$$

where in the last step we have introduced a Feynman parameter and used

$$
\begin{aligned}
&\frac{1}{\left(k^2-m_0^2+i\epsilon\right)\left((p-k)^2-\mu^2+i\epsilon\right)} \\
&= \int_0^1 dx\,\frac{1}{\left[k^2-2x\,k\cdot p+xp^2-x\mu^2-(1-x)m_0^2+i\epsilon\right]^2}
\end{aligned} \tag{190}
$$

This results has to be regularized: we have adopted dimensional regularization. We first perform a Wick rotation by letting $l_0 \to il_0$ so that l^2 becomes a Euclidean product and we can perform the integral in spherical coordinates. We get

$$
\int \frac{d^4l}{(2\pi)^4}\frac{-2x\,\not{p}+4m_0}{\left[l^2-\Delta+i\epsilon\right]^2} \to i\int \frac{d^4l}{(2\pi)^4}\frac{-2x\,\not{p}+4m_0}{\left[l^2+\Delta\right]^2} \tag{191}
$$

and repeating the calculatione above in N dimensions we get

$$
i\int \frac{d^4l}{(2\pi)^4}\frac{-2x\,\not{p}+4m_0}{\left[l^2+\Delta\right]^2} \to i\int \frac{d^Nl}{(2\pi)^N}\frac{-(N-2)\,x\,\not{p}+Nm_0}{\left[l^2+\Delta\right]^2} \tag{192}
$$

The integral can be computed using

$$
i\int \frac{d^Nl}{(2\pi)^N}\frac{1}{\left[l^2+\Delta\right]^2} = \frac{i}{(4\pi)^2}\frac{\Gamma\left(2-\frac{N}{2}\right)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{N}{2}} \tag{193}
$$

which in the $\bar{M}S$ scheme becomes

$$
\frac{i}{(4\pi)^2}\frac{\Gamma\left(2-\frac{N}{2}\right)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{N}{2}} \to \frac{i}{(4\pi)^2}\left(-\log\frac{\Delta}{M^2}\right). \tag{194}
$$

So going back to 4 dimensions we finally get

$$\Sigma_2(p) = \frac{g^2\left(t^A t^A\right)}{(4\pi)^2}\int_0^1 dx\,\log\frac{M^2}{-x(1-x)p^2 + x\mu^2 + (1-x)m_0^2} \left(4m_0 - 2x\,\not{p}\right). \tag{195}$$

We finally obtain

$$\delta Z_2 = \left[\frac{d\Sigma_2}{d\,\not{p}}\right]_{\not{p}=m} = \frac{g^2\left(t^A t^A\right)}{(4\pi)^2}\int_0^1 dx \left(-2x\log\frac{M^2}{(1-x)^2 m^2 + x\mu^2} + 2x(1-x)\frac{(4-2x)m^2}{(1-x)^2 m^2 + x\mu^2}\right). \tag{196}$$

In this subsection we have calculated the field strength renormalization, required to perform cross section computations at NLO.

6.7 Order α_s IR divergent contribution to the total cross section

We now have all the elements needed in order to find the total IR divergent contribution to the total cross section for the process $b\bar{b} \to H$ up to order α_s.

First we write the $O(\alpha_s)$ IR divergent contribution to the amplitude. The IR divergent part of the amplitude previously calculated is given by

$$iA_1^{IR} = \left(-g^2\right)\left(t^A t^A\right)\frac{m}{v}\int_0^1 dxdydz\,\delta(x+y+z-1)\; 2\int\frac{d^4l}{(2\pi)^4}\frac{\left[4m^2 z(z-1) + 4m^2 - 4y(y-1)q^2 - 2z(1-2y)q^2\right]}{\left[l^2 - \Delta + i\epsilon\right]^3}\bar{v}(p')\,u(p). \tag{197}$$

Performing a Wick rotation we get

$$\int\frac{d^4l}{(2\pi)^4}\frac{1}{\left[l^2-\Delta+i\epsilon\right]^3} \to \frac{i}{(-1)^3(2\pi)^4}\int d^4l\,\frac{1}{\left[l^2+\Delta\right]^3} = \frac{-i}{(4\pi)^2}\frac{1}{2}\frac{1}{\Delta}, \tag{198}$$

so that we obtain

$$A_1^{IR} = g^2\left(t^A t^A\right)\frac{m}{v}\int_0^1 dxdydz\,\delta(x+y+z-1)\; \frac{\left[4m^2 z(z-1) + 4m^2 - 4y(y-1)q^2 - 2z(1-2y)q^2\right]}{m^2(1-z)^2 - xyq^2}\bar{v}(p')\,u(p). \tag{199}$$

In order to get the total $O(\alpha_s)$ contribution to the amplitude coming from the $b\bar{b} \to H$ subprocess that we have to calculate, we find for the IR divergent part

$$A_1^{IR} + (\delta Z_2) A_0 \equiv A_{virtual}^{(1)} = \frac{g^2 \left(t^A t^A\right)}{(4\pi)^2} \frac{m}{v} \bar{v}(p') u(p)$$
$$\left\{ \int_0^1 dxdydz\, \delta(x+y+z-1) \frac{\left[4m^2 z(z-1) + 4m^2 - 4y(y-1) q^2 - 2z(1-2y) q^2\right]}{m^2 (1-z)^2 - xyq^2 + \mu^2 z} \right.$$
$$\left. - \int_0^1 dx \left(-2x \log \frac{M^2}{(1-x)^2 m^2 + x\mu^2} + 2x(1-x) \frac{(4-2x) m^2}{(1-x)^2 m^2 + x\mu^2} \right) \right\}, \tag{200}$$

where we have introduces a gluon mass μ to regularize the results. We now use the identity

$$\delta Z_2 = \frac{g^2 \left(t^A t^A\right)}{(4\pi)^2} \int_0^1 dx \left(-2x \log \frac{M^2}{(1-x)^2 m^2 + x\mu^2} + 2x(1-x) \frac{(4-2x) m^2}{(1-x)^2 m^2 + x\mu^2} \right)$$
$$= -\frac{g^2 \left(t^A t^A\right)}{(4\pi)^2} \int_0^1 dxdydz \delta(x+y+z-1)\, 2 \left[\log \frac{zM^2}{(1-z)^2 m^2 + z\mu^2} + \frac{\left(1-4z+z^2\right) m^2}{(1-z)^2 m^2 + z\mu^2} \right]$$
$$= -\frac{g^2 \left(t^A t^A\right)}{(4\pi)^2} \int_0^1 dz (1-z)\, 2 \left[\log \frac{zM^2}{(1-z)^2 m^2 + z\mu^2} + \frac{\left(1-4z+z^2\right) m^2}{(1-z)^2 m^2 + z\mu^2} \right]. \tag{201}$$

The second line of this last equation is the one we will insert in Eq. (200) in the next step, while from the third line we see that when $\mu \to 0$ the logarithmic term is not a leading one because $\lim_{x\to 0} \frac{\log x}{x} = 0$. Substituting in Eq. (200) and taking only the dominant part in the limit $\mu \to 0$ we find

$$A_{virtual}^{(1)} = \frac{g^2 \left(t^A t^A\right)}{(4\pi)^2} \frac{m}{v} \bar{v}(p') u(p)$$
$$\left\{ \int_0^1 dxdydz\, \delta(x+y+z-1) \frac{\left[4m^2 z(z-1) + 4m^2 - 4y(y-1) q^2 - 2z(1-2y) q^2\right]}{m^2 (1-z)^2 - xyq^2 + \mu^2 z} \right.$$
$$\left. + 2 \frac{\left(1-4z+z^2\right) m^2}{(1-z)^2 m^2 + z\mu^2} \right\}. \tag{202}$$

We are interested in the divergent part of this expression. The divergence occurs when the Feynman parameters are $z \sim 1$, $x \sim y \sim 0$ so in this region we can set $z=1$, $x=y=0$ in the numerator and $z=1$ also in the μ^2 terms in the denominator getting

$$A^{(1)}_{virtual} = \frac{g^2\left(t^At^A\right)}{(4\pi)^2}\frac{m}{v}\,\bar{v}\left(p'\right)u\left(p\right)$$

$$\int_0^1 dxdydz\,\delta\left(x+y+z-1\right)\left[\frac{4m^2-2q^2}{m^2\left(1-z\right)^2-xyq^2+\mu^2}-\frac{4m^2}{\left(1-z\right)^2m^2+\mu^2}\right]$$

$$= \frac{g^2\left(t^At^A\right)}{(4\pi)^2}\frac{m}{v}\,\bar{v}\left(p'\right)u\left(p\right)\int_0^1 dz\int_0^{1-z}dy\left[\frac{4m^2-2q^2}{m^2\left(1-z\right)^2-y\left(1-y-z\right)q^2+\mu^2}\right.$$
$$\left.-\frac{4m^2}{\left(1-z\right)^2m^2+\mu^2}\right]$$

$$= -\frac{g^2\left(t^At^A\right)}{2\left(4\pi\right)^2}\frac{m}{v}\,\bar{v}\left(p'\right)u\left(p\right)\int_0^1 d\psi\int_0^1 d\left(w^2\right)\left[\frac{-4m^2+2q^2}{\left[m^2-q^2\psi\left(1-\psi\right)\right]w^2+\mu^2}-\frac{-4m^2}{w^2m^2+\mu^2}\right]$$

$$= -\frac{g^2\left(t^At^A\right)}{2\left(4\pi\right)^2}\frac{m}{v}\,\bar{v}\left(p'\right)u\left(p\right)\int_0^1 d\psi\left[\frac{-4m^2+2q^2}{m^2-q^2\psi\left(1-\psi\right)}\log\frac{m^2-q^2\psi\left(1-\psi\right)}{\mu^2}+\log\frac{m^2}{\mu^2}\right], \tag{203}$$

where we have defined

$$\begin{aligned} y &= \left(1-z\right)\psi \\ w &= \left(1-z\right), \end{aligned} \tag{204}$$

and in the arguments of the logarithms we have dropped the terms of order 1 which are irrelevant in the limit $\mu\to 0$. We notice that in this limit

$$\log\frac{m^2-q^2\psi\left(1-\psi\right)}{\mu^2}\approx\log\frac{m^2}{\mu^2},$$

so that we obtain

$$A^{(1)}_{virtual} =$$

$$\frac{g^2\left(t^At^A\right)}{2\left(4\pi\right)^2}\frac{m}{v}\,\bar{v}\left(p'\right)u\left(p\right)4\left[\int_0^1 d\psi\frac{m^2-\frac{q^2}{2}}{m^2-q^2\psi\left(1-\psi\right)}-1\right]\log\frac{m^2}{\mu^2} \tag{205}$$

$$= \frac{\alpha_s\left(t^At^A\right)}{2\pi}\frac{m}{v}\,\bar{v}\left(p'\right)u\left(p\right)f_{IR}\left(q^2\right)\log\frac{m^2}{\mu^2},$$

where we have used $\alpha_s = \frac{g^2}{4\pi}$ and we have defined

$$f_{IR}\left(q^2\right) = \int_0^1 d\psi \frac{m^2 - \frac{q^2}{2}}{m^2 - q^2\psi\left(1-\psi\right)} - 1. \tag{206}$$

Now that we have worked out the total IR divergent contribution to the amplitude, we are ready to determine the corresponding total cross section. Starting with

$$A = A_0 + A_1 + \left(\delta Z_2\right) A_0 = A_0 + A_1^{UV} + A_1^{IR} + \left(\delta Z_2\right) A_0 \tag{207}$$

we will have

$$\begin{aligned} \mid A \mid^2 &= \left(A_0 + A_1^{UV} + A^{(1)}_{virtual}\right)\left(\bar{A}_0 + \bar{A}_1^{\,UV} + \bar{A}^{(1)}_{virtual}\right) \\ &= \mid A_0 \mid^2 + 2\left(A_1^{UV}\bar{A}_0\right) + 2\left(A^{(1)}_{virtual}\bar{A}_0\right). \end{aligned} \tag{208}$$

Therefore, as for the contribution coming from IR divergent part we have, averaging on the initial polarization and colors and adding the flux factor and phase space integral

$$\frac{1}{4}\frac{1}{9}\sum_{colors}\sum_{pol} 2\left(A^{(1)}_{virtual}\bar{A}_0\right) = -\frac{\alpha_s}{\pi}\sigma_0 f_{IR}\left(q^2\right)\log\frac{m^2}{\mu^2}. \tag{209}$$

6.8 Order α_s UV divergent contribution to the total cross section

We will now determine the contribution to the total cross section up to order α_s coming from the UV divergent part of the amplitude for the process $b\bar{b} \to H$. One particular feature of this section is that we have to renormalize the Yukawa coupling, adding therefore a new counterterm. This is not necessary in other similar calculations, such as the Drell-Yan process, due to a Ward identity which cancels the ultraviolet divergences.

The UV divergent part of the 1-loop amplitude of Fig. (7) is given by

$$\begin{aligned} iA_1^{UV} = -g^2\left(t^A t^A\right)\left(\frac{m}{v}\right)\int_0^1 dxdydz\,\delta\left(x+y+z-1\right) \\ \int\frac{d^4 l}{\left(2\pi\right)^4}\frac{8l^2}{\left[l^2-\Delta+i\epsilon\right]^3}\bar{v}\left(p'\right)u\left(p\right). \end{aligned} \tag{210}$$

We proceed using dimensional regularization. Shifting in N dimensions and performing a Wick rotation we have in general

$$\begin{aligned} \int\frac{d^N l}{\left(2\pi\right)^N}\frac{l^2}{\left[l^2-\Delta+i\epsilon\right]^n} &\to i\int\frac{d^N l}{\left(2\pi\right)^N}\frac{l^2}{\left[l^2+\Delta\right]^n} \\ &= \frac{i}{\left(4\pi\right)^{\frac{N}{2}}}\frac{N}{2}\frac{\Gamma\left(n-\frac{N}{2}-1\right)}{\Gamma\left(n\right)}\left(\frac{1}{\Delta}\right)^{n-\frac{N}{2}-1}. \end{aligned} \tag{211}$$

Furthermore, calculating the numerator in N dimension we get

$$
\begin{aligned}
&-\gamma^{\nu}\not{k}\not{k}'\gamma_{\nu}-m\gamma^{\nu}\not{k}'\gamma_{\nu}+m\gamma^{\nu}\not{k}\gamma_{\nu}+m^{2}\gamma^{\nu}\gamma_{\nu}\\
&=-4\left(k\cdot k'\right)+\left(4-N\right)\not{k}\cdot\not{k}'+2m\left(N-2\right)\not{k}'-2m\left(N-2\right)\not{k}+Nm^{2}\\
&=-4k\cdot\left(q-k\right)+\left(4-N\right)\not{k}\cdot\left(\not{q}-\not{k}\right)+2m\left(N-2\right)\left(\not{q}-\not{k}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad-2m\left(N-2\right)\not{k}+Nm^{2}\\
&=N\left(k\cdot k\right)+..
\end{aligned}
\tag{212}
$$

so that we obtain

$$
\begin{aligned}
A_1^{UV}=-g^{2}\left(t^{A}t^{A}\right)\left(\frac{m}{v}\right)\int_{0}^{1}dxdydz\,\delta\left(x+y+z-1\right)\\
\frac{2N}{\left(4\pi\right)^{\frac{N}{2}}}\frac{\Gamma\left(2-\frac{N}{2}\right)}{\Gamma\left(3\right)}\left(\frac{1}{\Delta}\right)^{2-\frac{N}{2}}\bar{v}\left(p'\right)u\left(p\right).
\end{aligned}
\tag{213}
$$

Finally, in the $\bar{MS}$ scheme and restoring $N=4$ we get

$$
\begin{aligned}
A_1^{UV}&=-g^{2}\left(t^{A}t^{A}\right)\left(\frac{m}{v}\right)\int_{0}^{1}dxdydz\,\delta\left(x+y+z-1\right)\\
&\qquad\frac{8}{\left(4\pi\right)^{2}}\left(-\log\frac{\Delta}{M^{2}}\right)\bar{v}\left(p'\right)u\left(p\right)\\
&=\alpha_{s}\frac{t^{A}t^{A}}{4\pi}\frac{m}{v}\int_{0}^{1}\delta\left(x+y+z-1\right)\log\frac{m^{2}\left(1-z\right)^{2}-xyq^{2}}{M^{2}}\bar{v}\left(p'\right)u\left(p\right)\\
&=\alpha_{s}\frac{t^{A}t^{A}}{4\pi}\frac{m}{v}\left[2D\right]\bar{v}\left(p'\right)u\left(p\right),
\end{aligned}
\tag{214}
$$

with

$$
\begin{aligned}
D&=\frac{1}{2}\int_{0}^{1}\delta\left(x+y+z-1\right)\log\frac{m^{2}\left(1-z\right)^{2}-xyq^{2}}{M^{2}}\\
&=\int_{0}^{1}dx\int_{0}^{1-x}dy\,\log\frac{m^{2}\left(x+y\right)^{2}-xyq^{2}}{M^{2}}.
\end{aligned}
\tag{215}
$$

This is the renormalized UV divergent contribution to the amplitude. In order to obtain the corresponding term for the total cross section, we have to work out the interference term between the two diagrams of Fig. (7), as we have already done to calculate the IR divergent part. Therefore we have

$$
2\frac{1}{9}\sum_{colors}\frac{1}{4}\sum_{pol}A_1^{UV}\bar{A}_0=-\frac{\alpha_{s}}{\pi}\sigma_{0}\left[D\right]
\tag{216}
$$

where D depends on the renormalization scale M^2.

To sum up, the calculations done in the last subsections, which are summarized in Fig. (9), lead to the total cross section

$$\hat{\sigma}_{virtual} = \sigma_0 \left(1 - \frac{\alpha_s}{\pi} f_{IR}\left(q^2\right) \log\frac{m^2}{\mu^2} - \frac{\alpha_s}{\pi}\left[D\right]\right), \tag{217}$$

which is still IR divergent. We now have to calculate and to add to this result the contribution of the same order coming from the real emissions.

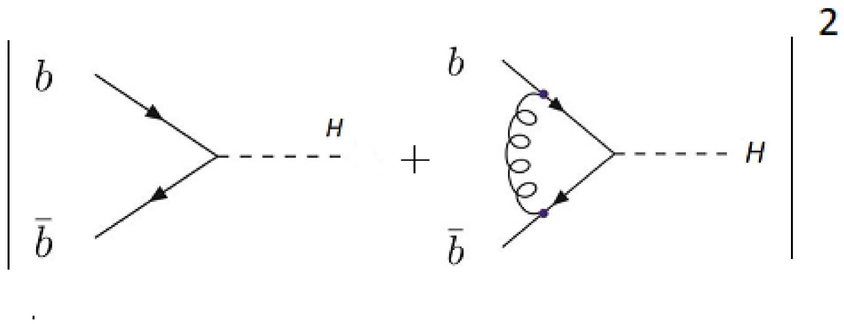

Figure 9: Feynman diagrams for the square amplitude up to order α_s for the process $b\bar{b} \to H$

6.9 Real emission

In this subsection we calculate the contribution to the total cross section coming from the real emissions of a gluon, as shown in the Feynman diagrams of Fig. (10).

The amplitude for real emission of a gluon in the process $b\bar{b} \to Hg$ is given by

$$iA_{real} = \bar{v}\left(p'\right)\left(-i\frac{m}{v}\right)$$
$$\left(\left(-igt^A\gamma^\mu\epsilon_\mu\left(k\right)\right)\frac{i\left(-\not{p}' + \not{k} + m\right)}{\left(\left(p'-k\right)^2 - m^2\right)} + \frac{i\left(\not{p} - \not{k} + m\right)}{\left(\left(p-k\right)^2 - m^2\right)}\left(-igt^A\gamma^\nu\epsilon_\nu\left(k\right)\right)\right)u\left(p\right). \tag{218}$$

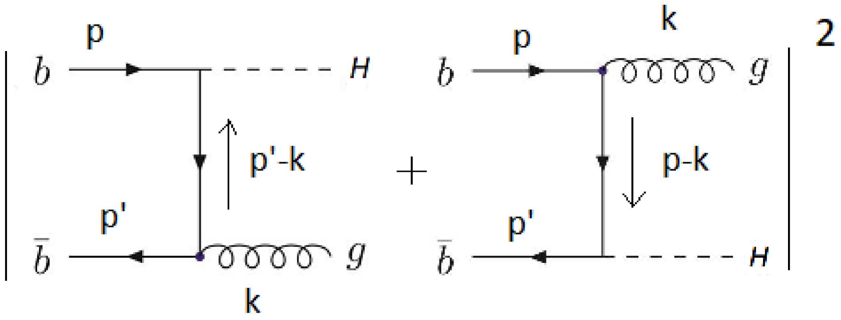

Figure 10: Feynman diagrams for the square amplitude for the process $b\bar{b} \to Hg$

In order to simplify the equations above we use the identity

$$\begin{aligned}
(\not{p} + m)\,\gamma^{\nu}\epsilon_{\nu} &= (\gamma^{\mu}\gamma^{\nu}p_{\mu}\epsilon_{\nu} + m\gamma^{\nu}\epsilon_{\nu}) \\
&= (p^{\mu}\epsilon_{\mu} + m\gamma^{\nu}\epsilon_{\nu}) \\
&= (2p^{\mu}\epsilon_{\mu} - p^{\mu}\epsilon_{\mu} + m\gamma^{\nu}\epsilon_{\nu}) \\
&= (2p^{\mu}\epsilon_{\mu} - \gamma^{\nu}\gamma^{\mu}p^{\mu}\epsilon_{\nu} + m\gamma^{\nu}\epsilon_{\nu}) \\
&= (2p^{\mu}\epsilon_{\mu} + \gamma^{\nu}\epsilon_{\nu}\,(-\not{p} + m))\,,
\end{aligned} \tag{219}$$

thanks to which, using $\not{u}\,(p) = mu\,(p)$

$$(\not{p} + m)\,\gamma^{\nu}\epsilon_{\nu}u\,(p) = 2p^{\mu}\epsilon_{\mu}u\,(p)\,. \tag{220}$$

In the same way

$$(-\not{p}' + m)\,\gamma^{\mu}\epsilon_{\mu} = (-2p'^{\mu}\epsilon_{\mu} + \gamma^{\nu}\epsilon_{\nu}\,(\not{p}' + m))\,, \tag{221}$$

from which using $\bar{v}\,(p')\,\not{p}' = -\bar{v}\,(p')\,m$

$$\bar{v}\,(p')\,(-\not{p}' + m)\,\gamma^{\mu}\epsilon_{\mu} = -2\bar{v}\,(p')\,p'^{\mu}\epsilon_{\mu}. \tag{222}$$

Therefore we have

$$iA_{real} = -igt^{A}\left(\frac{m}{v}\right)\bar{v}\,(p')\,u\,(p)\left(\frac{-p'^{\mu}}{(-p')\cdot k} - \frac{p^{\mu}}{p\cdot k}\right)\epsilon_{\mu}, \tag{223}$$

from which, taking the spin and color average result and multiplying for the flux factor

$$\sum_{colors}\frac{1}{4}\sum_{pol}\mid A_{real}\mid^{2}=g^{2}\sigma_{0}\sum_{\lambda}\epsilon_{\mu}\bar{\epsilon}_{\nu}\left(\frac{-p'^{\mu}}{(-p')\cdot k}-\frac{p^{\mu}}{p\cdot k}\right)\left(\frac{-p'^{\nu}}{(-p')\cdot k}-\frac{p^{\nu}}{p\cdot k}\right)$$
$$=g^{2}\sigma_{0}\left(\frac{2\left(-p'\cdot p\right)}{\left(-p'\cdot k\right)\left(p\cdot k\right)}-\frac{m^{2}}{\left(-p'\cdot k\right)^{2}}-\frac{m^{2}}{\left(p\cdot k\right)^{2}}\right). \tag{224}$$

In order to get the cross section in the soft limit we provide the contribution of the phase space of the soft gluon

$$\sigma_{real}=g^{2}\sigma_{0}\int\frac{d^{3}k}{\left(2\pi\right)^{3}}\frac{1}{2\mid\vec{k}\mid}\left(\frac{2\left(-p'\cdot p\right)}{\left(-p'\cdot k\right)\left(p\cdot k\right)}-\frac{m^{2}}{\left(-p'\cdot k\right)^{2}}-\frac{m^{2}}{\left(p\cdot k\right)^{2}}\right)$$

$$=g^{2}\sigma_{0}\int\frac{\mid\vec{k}\mid^{2}d\mid\vec{k}\mid d\Omega}{\left(2\pi\right)^{3}}\frac{1}{2\mid\vec{k}\mid^{3}}\left(\frac{2\left(-p'\cdot p\right)}{\left(-p'\cdot\hat{k}\right)\left(p\cdot\hat{k}\right)}-\frac{m^{2}}{\left(-p'\cdot\hat{k}\right)^{2}}-\frac{m^{2}}{\left(p\cdot\hat{k}\right)^{2}}\right)$$
$$=\frac{g^{2}\sigma_{0}}{\left(2\pi\right)^{3}}\frac{4\pi}{2}\int\frac{d\mid\vec{k}\mid}{\mid\vec{k}\mid}\int\frac{d\Omega}{4\pi}\left(\frac{2\left(-p'\cdot p\right)}{\left(-p'\cdot\hat{k}\right)\left(p\cdot\hat{k}\right)}-\frac{m^{2}}{\left(-p'\cdot\hat{k}\right)^{2}}-\frac{m^{2}}{\left(p\cdot\hat{k}\right)^{2}}\right). \tag{225}$$

We now use the identities

$$\int\frac{d\Omega}{4\pi}\frac{1}{\left(\hat{k}\cdot p\right)^{2}}=\frac{1}{2}\int_{-1}^{1}dcos\theta\frac{1}{\left(p^{0}-p\cos\theta\right)^{2}}=\frac{1}{p^{2}}=\frac{1}{m^{2}}$$
$$\int\frac{d\Omega}{4\pi}\frac{1}{\left(-p'\cdot\hat{k}\right)^{2}}=\frac{1}{2}\int_{-1}^{1}dcos\theta\frac{1}{\left(p'^{0}-p'\cos\theta\right)^{2}}=\frac{1}{p'^{2}}=\frac{1}{m^{2}}, \tag{226}$$

and

$$q=(p+p')\rightarrow-2p\cdot p'=2m^{2}-q^{2}$$

$$\int\frac{d\Omega}{4\pi}\frac{1}{\left(-p'\cdot\hat{k}\right)\left(p\cdot\hat{k}\right)}=\int_{0}^{1}d\psi\int\frac{d\Omega}{4\pi}\frac{1}{\left[\psi\left(-p'\cdot\hat{k}\right)+\left(1-\psi\right)\hat{k}\cdot p\right]^{2}} \tag{227}$$
$$=\int_{0}^{1}d\psi\frac{1}{\left[-\psi p'+\left(1-\psi\right)p\right]^{2}}=\int_{0}^{1}d\psi\frac{1}{m^{2}-\psi\left(1-\psi\right)q^{2}},$$

so that

$$\int \frac{d\Omega}{4\pi} \left(\frac{2\left(-p' \cdot p\right)}{\left(-p' \cdot \hat{k}\right)\left(p \cdot \hat{k}\right)} - \frac{m^2}{\left(-p' \cdot \hat{k}\right)^2} - \frac{m^2}{\left(p \cdot \hat{k}\right)^2} \right)$$
$$= \int_0^1 d\psi \frac{2m^2 - q^2}{m^2 - \psi\left(1-\psi\right) q^2} - 2 = 2 f_{IR}\left(q^2\right). \tag{228}$$

Finally we have

$$\sigma_{real} = \frac{\alpha_s}{\pi}\sigma_0 \frac{1}{2}\left(\log\frac{E_f^2}{\mu^2}\right) 2 f_{IR}\left(q^2\right) = \frac{\alpha_s}{\pi}\sigma_0 \left(\log\frac{E_f^2}{\mu^2}\right) f_{IR}\left(q^2\right), \tag{229}$$

where E_f^2 is some minimum energy down to which the gluons can be detected.

After computing both the virtual and real corrections up to order α_s to the subprocess $b\bar{b} \to H$, we can now write down the total cross section: the separate virtual and real cross sections are divergent, but their sum is independent of μ^2 and therefore finite. Adding the two previous results we obtain

$$\hat{\sigma}_{b\bar{b}}^{(4)} = \sigma_{virtual} + \sigma_{real}$$
$$= \sigma_0 - \sigma_0 \frac{\alpha_s}{\pi} f_{IR}\left(q^2\right) \log\frac{m^2}{\mu^2} - \sigma_0 \frac{\alpha_s}{\pi}\left[D\right] + \frac{\alpha_s}{\pi}\sigma_0 \left(\log\frac{E_f^2}{\mu^2}\right) f_{IR}\left(q^2\right) \tag{230}$$
$$= \sigma_0 \left(1 + \frac{\alpha_s}{\pi} f_{IR}\left(q^2\right) \log\frac{E_f^2}{m^2} - \frac{\alpha_s}{\pi}\left[D\right]\right).$$

We see that the IR divergences cancel between the real and virtual correction, leaving a result which is both IR and UV finite. We notice the presence of the large logarithm of E_f^2/m^2, which is the collinear divergence regularized by the heavy quark mass. Despite the simplicity of this result, we have not retained the full dependence on q^2, so that we are not able to reconstruct the precise coefficient of this logarithmic term, whose precise expression is given by the results computed in dimensional regularization of the previous subsection.

6.10 The subprocess $bg \to Hb$

In this subsection we calculate the cross section for the subprocess $bg \to bH$, whose diagrams are shown in Fig. (11), which is the other result needed for the implementation of the generalized FONLL results of the previous section. As for the subprocess $\bar{b}g \to \bar{b}H$, its diagrams are topologically equivalent to those of fig.(11), and therefore the corresponding cross sections are the same.

We define

$$q' = p_2 - p_4 = p_3 - p_1 \to q^2 = m^2 - 2p_2 \cdot p_4$$
$$q'' = p_1 + p_2 = p_3 + p_4 \to q^2 = m^2 + 2p_2 \cdot p_1 \tag{231}$$

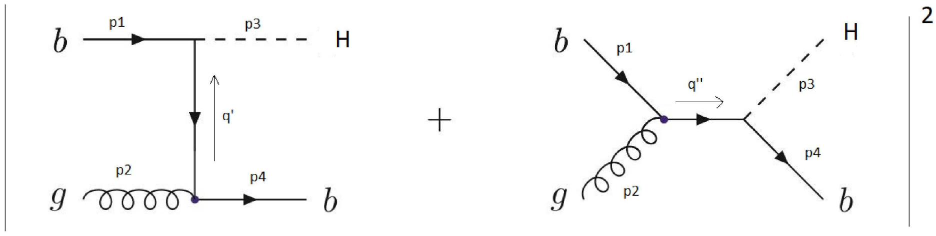

Figure 11: Feynman diagrams for the square amplitude for the process $b \to Hb$

The amplitude for the process is given by the sum

$$iM = iM_1 + iM_2, \tag{232}$$

therefore the square amplitude is

$$\mid M \mid^2 = \mid M_1 \mid^2 + \mid M_2 \mid^2 + 2Re\left(M_1 M_2^*\right). \tag{233}$$

Using the QCD Feynman rules we obtain

$$\begin{aligned} iM_1 &= \bar{u}\left(p_4\right)\left(-igt^A\gamma^\mu\right)\epsilon_\mu\left(p_2\right)\frac{i\left(\not{p}_4 - \not{p}_2 + m\right)}{\left(q'\right)^2 - m^2 + i\epsilon}u\left(p_1\right)\left(-i\frac{m}{v}\right) \\ &= -ig\,t^A\frac{m}{v}\frac{\epsilon_\mu\left(p_2\right)}{\left(-2p_2\cdot p_4\right)}\bar{u}\left(p_4\right)\gamma^\mu\left(\not{p}_4 - \not{p}_2 + m\right)u\left(p_1\right) \end{aligned} \tag{234}$$

$$\begin{aligned} iM_2 &= \bar{u}\left(p_4\right)\left(-i\frac{m}{v}\right)\frac{i\left(\not{p}_1 + \not{p}_2 + m\right)}{\left(\left(q''\right)^2 - m^2 + i\epsilon\right)}\epsilon_\mu\left(p_2\right)\left(-igt^A\gamma^\mu\right)u\left(p_1\right) \\ &= -igt^A\frac{m}{v}\frac{\bar{u}\left(p_4\right)\left(\not{p}_1 + \not{p}_2 + m\right)\epsilon_\mu\left(p_2\right)\gamma^\mu u\left(p_1\right)}{2\left(p_1\cdot p_2\right)}. \end{aligned} \tag{235}$$

From these expressions we get the color and spin average square amplitudes, which are respectively

$$\begin{aligned} \sum_{colors}\frac{1}{4}\sum_{pol}\mid M_1\mid^2 &= -g^2\left(\frac{4}{3}\right)\left(\frac{m}{v}\right)^2\frac{1}{\left(-2p_2\cdot p_4\right)^2} \\ &\frac{1}{4}Tr\left[\gamma_\mu\left(\not{p}_4 + m\right)\gamma^\mu\left(\not{p}_4 - \not{p}_2 + m\right)\left(\not{p}_1 + m\right)\left(\not{p}_4 - \not{p}_2 + m\right)\right] \end{aligned} \tag{236}$$

$$\begin{aligned} \sum_{colors}\frac{1}{4}\sum_{pol}\mid M_2\mid^2 &= -g^2\left(\frac{4}{3}\right)\left(\frac{m}{v}\right)^2\frac{1}{16\left(p_1\cdot p_2\right)^2} \\ &Tr\left[\left(\not{p}_4 + m\right)\left(\not{p}_1 + \not{p}_2 + m\right)\gamma^\mu\left(\not{p}_1 + m\right)\gamma_\mu\left(\not{p}_1 + \not{p}_2 + m\right)\right], \end{aligned} \tag{237}$$

where we have used

$$\sum_{pol}\epsilon_\mu\epsilon_\nu^* = -g_{\mu\nu}, \tag{238}$$

while for the interference term we have

$$2Re\sum_{colors}\frac{1}{4}\sum_{pol}M_1M_2^* = g^2\left(\frac{m}{v}\right)^2\frac{1}{6}\frac{1}{(p_2\cdot p_4)(p_2\cdot p_1)}$$
$$Tr\left[(\not p_4+m)\gamma^\mu(\not p_4-\not p_2+m)(\not p_1+m)\gamma_\mu(\not p_2+\not p_1+m)\right]. \tag{239}$$

Using the identities

$$\begin{aligned}
&\gamma^\mu\gamma_\mu = 4\\
&\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu\\
&\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}
\end{aligned} \tag{240}$$

we reduce the traces above to known traces of gamma matrix and eventually we obtain

$$\begin{aligned}
&Tr\left[\gamma_\mu(\not p_4+m)\gamma^\mu(\not p_4-\not p_2+m)(\not p_1+m)(\not p_4-\not p_2+m)\right]\\
&= 16\left[-(p_4\cdot p_2)(p_2\cdot p_1)+m^2(p_4\cdot p_1)-m^2(p_2\cdot p_1)-m^2(p_2\cdot p_4)+m^4\right]\\
&\\
&Tr\left[(\not p_4+m)(\not p_1+\not p_2+m)\gamma^\mu(\not p_1+m)\gamma_\mu(\not p_1+\not p_2+m)\right]\\
&= 16\left[-(p_4\cdot p_2)(p_1\cdot p_2)+m^2(p_4\cdot p_2)+m^2(p_4\cdot p_1)+m^2(p_2\cdot p_1)+m^4\right]\\
&\\
&Tr\left[(\not p_4+m)\gamma^\mu(\not p_4-\not p_2+m)(\not p_1+m)\gamma_\mu(\not p_2+\not p_1+m)\right]\\
&= 16\Big[(p_4\cdot p_1)(p_2\cdot p_4)-(p_2\cdot p_4)(p_2\cdot p_1)+(p_1\cdot p_2)(p_4\cdot p_1)\\
&\qquad -(p_1\cdot p_2)^2+m^2(p_2\cdot p_4)\Big].
\end{aligned} \tag{241}$$

Therefore we get

$$\begin{aligned}
&\sum_{colors}\frac{1}{4}\sum_{pol}\mid M\mid^2 = g^2\left(\frac{4}{3}\right)\left(\frac{m}{v}\right)^2\\
&\left[\frac{2(p_4\cdot p_1)-(p_2\cdot p_1)+m^2}{(p_2\cdot p_4)}-\frac{m^2(p_4\cdot p_1)-m^2(p_2\cdot p_1)+m^4}{(p_2\cdot p_4)^2}\right.\\
&\left.+\frac{2(p_4\cdot p_1)-(p_4\cdot p_2)+3m^2}{(p_1\cdot p_2)}+\frac{m^2(p_4\cdot p_2)+m^2(p_4\cdot p_1)+m^4}{(p_1\cdot p_2)^2}-2\right].
\end{aligned} \tag{242}$$

Now we have to add the phase space. Unlike in the previous case of real emission of a gluon from b quark, now we have no IR divergences coming from the phase space because $E_4 = (m_4^2+\mid\vec{p_4}\mid^2)^{\frac{1}{2}} \neq\mid\vec{p_4}\mid$ so that when $\mid\vec{p_4}\mid\to 0$ we still have no divergences. We get

$$
\begin{aligned}
d\phi_2 &= \frac{d^3p_4}{(2\pi)^3 2E_4}\frac{d^3p_3}{(2\pi)^3 2E_3}(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\\
&= \frac{1}{(2\pi)^2}\frac{1}{4E_3E_4}\delta(E_1+E_2-E_3-E_4)\mid \vec{p_4}\mid^2 d\mid \vec{p_4}\mid d\Omega\\
&= \frac{1}{(2\pi)^2}\frac{1}{4(E_1+E_2-E_4)E_4}\sum_{\pm}\frac{\delta\left(\mid\vec{p_4}\mid-\mid\vec{p_4}\mid_0^{\pm}\right)}{\mid f'\left(\mid\vec{p_4}\mid_0^{\pm}\right)\mid}\mid\vec{p_4}\mid^2 d\mid\vec{p_4}\mid d\Omega,
\end{aligned}
\tag{243}
$$

where

$$
\begin{aligned}
f(\mid\vec{p_4}\mid) &= E_1+E_2-E_3-E_4\\
&= E_1+E_2-\sqrt{m_3^2+\mid\vec{p_3}\mid^2}-\sqrt{m_4^2+\mid\vec{p_4}\mid^2}\\
&= E_1+E_2-\sqrt{m_3^2+\mid\vec{p_1}+\vec{p_2}-\vec{p_4}\mid^2}-\sqrt{m_4^2+\mid\vec{p_4}\mid^2}
\end{aligned}
$$

$$
\begin{aligned}
&f\left(\mid\vec{p_4}\mid_0^{\pm}\right)=0\\
&\rightarrow\mid p_4\mid_0^{\pm} = \frac{1}{4(E_1+E_2)^2}\Bigg[cos\theta\left(\mid\vec{p_2}\mid-\mid\vec{p_1}\mid\right)\\
&\pm\Bigg(\cos^2\theta\left(\mid\vec{p_2}\mid-\mid\vec{p_1}\mid\right)^2-4(E_1+E_2)^4\left(4m_4^2+1\right)\\
&\quad+4(E_1+E_2)^2\left(m_3^2-m_4^2+\left(\mid\vec{p_1}\mid-\mid\vec{p_2}\mid\right)^2\right)\Bigg)^{\frac{1}{2}}\Bigg].
\end{aligned}
\tag{244}
$$

The cross section is given by

$$
\begin{aligned}
\hat{\sigma}_{bg} &= \frac{1}{4(p\cdot p')}\sum_{colors}\frac{1}{4}\sum_{pol}\mid M\mid^2 d\phi_2 = \frac{1}{4(p_1\cdot p_2)}\frac{1}{(2\pi)^2}\sum_{\pm}\frac{\delta\left(\mid\vec{p_4}\mid-\mid\vec{p_4}\mid_0^{\pm}\right)}{\mid f'\left(\mid\vec{p_4}\mid_0^{\pm}\right)\mid}\\
&g^2\left(\frac{4}{3}\right)\left(\frac{m}{v}\right)^2\frac{1}{4E_3E_4}\Bigg\{\frac{2(p_4\cdot p_1)-(p_2\cdot p_1)+m^2}{(p_2\cdot p_4)}-\frac{m^2(p_4\cdot p_1)-m^2(p_2\cdot p_1)+m^4}{(p_2\cdot p_4)^2}\\
&+\frac{2(p_4\cdot p_1)-(p_4\cdot p_2)+3m^2}{(p_1\cdot p_2)}+\frac{m^2(p_4\cdot p_2)+m^2(p_4\cdot p_1)+m^4}{(p_1\cdot p_2)^2}-2\Bigg\}\mid\vec{p_4}\mid^2 d\mid\vec{p_4}\mid d\Omega.
\end{aligned}
\tag{245}
$$

Defining θ as the angle between p_1 and p_4 the leading contribution to the cross section comes from the limit $\theta\rightarrow\pi$ as we see by writing explicitly

$$
\begin{aligned}
p_2\cdot p_4 &= E_2E_4-\vec{p_2}\cdot\vec{p_4} = E_2E_4+\mid\vec{p_2}\mid\mid\vec{p_4}\mid\cos\theta\\
&= \vec{p_2}\left(E_4+\mid\vec{p_4}\mid\cos\theta\right)
\end{aligned}
\tag{246}
$$

therefore in order to simplify the calculations we let $\cos\theta = -1$ also in the phase space, so that the dependence on θ remains only in the square amplitude. We now integrate with respect to $d\cos\theta$.

$$\begin{aligned}\int \frac{d\Omega}{4\pi}\frac{(p_4\cdot p_1)}{(p_2\cdot p_4)} &= \int d\cos\theta\Bigg\{-2\frac{\mid \vec{p_1}\mid}{\mid \vec{p_2}\mid}+\Bigg(2E_4E_1 \\ &\qquad + \frac{2\mid \vec{p_1}\mid}{\mid \vec{p_2}\mid}E_2E_4\Bigg)\frac{1}{E_2}\frac{1}{E_4+\mid \vec{p_4}\mid\cos\theta}\Bigg\} \\ &= -2\frac{\mid \vec{p_1}\mid}{\mid \vec{p_2}\mid}+\left(\frac{E_4E_1}{E_2}+\frac{\mid \vec{p_1}\mid}{\mid \vec{p_2}\mid}E_4\right)\log\left(1+\frac{2\mid \vec{p_4}\mid}{E_4-\mid \vec{p_4}\mid}\right) \\ &= -2\frac{\mid \vec{p_1}\mid}{\mid \vec{p_2}\mid}+A\left(p_1,p_2,p_4\right)\log\left(1+\frac{2\mid \vec{p_4}\mid}{E_4-\mid \vec{p_4}\mid}\right)\end{aligned} \tag{247}$$

$$\begin{aligned}\int \frac{d\Omega}{4\pi}\frac{(p_4\cdot p_1)}{(p_2\cdot p_4)^2} &= \int \frac{d\Omega}{4\pi}\frac{E_4E_1-\mid \vec{p_4}\mid\mid \vec{p_1}\mid\cos\theta}{(E_4E_2+\mid \vec{p_2}\mid\mid \vec{p_4}\mid\cos\theta)^2} \\ &= B\left(p_1,p_2,p_4\right),\end{aligned}$$

where $B\left(p_1,p_2,p_4\right)$ is explicitly found using

$$\int dx\frac{a+bx}{(c+dx^2)^2} = \frac{b}{2d^2}\log\frac{c^2+2cd+d^2}{c^2-2cd+d^2}+\left(a-\frac{cb}{d}\right)\frac{dx}{(c+dx)^2}. \tag{248}$$

Then we use

$$\int \frac{d\Omega}{(4\pi)}\frac{m^2-(p_2\cdot p_1)}{(p_2\cdot p_4)^2} = \frac{m^2-(p_2\cdot p_1)}{\mid \vec{p_2}\mid^2}\int \frac{d\Omega}{(4\pi)}\frac{1}{(\hat{p}_2\cdot p_4)^2} = \frac{m^2-(p_2\cdot p_1)}{\mid \vec{p_2}\mid^2}\frac{1}{m^2}$$

$$\begin{aligned}\int \frac{d\Omega}{(4\pi)}\frac{m^2-(p_2\cdot p_1)}{(p_2\cdot p_4)} &= \frac{m^2-(p_2\cdot p_1)}{2\mid \vec{p_2}\mid}\int_{-1}^{+1} d\cos\theta\frac{1}{E_4+\mid \vec{p_4}\mid\cos\theta} \\ &= \frac{m^2-(p_2\cdot p_1)}{2\mid \vec{p_2}\mid\mid \vec{p_4}\mid}\log\left(1+\frac{2\mid \vec{p_4}\mid}{E_4-\mid \vec{p_4}\mid}\right)\approx\frac{m^2-(p_2\cdot p_1)}{2\mid \vec{p_2}\mid\mid \vec{p_4}\mid}\log\left(4\frac{\mid \vec{p_4}\mid^2}{m^2}\right),\end{aligned} \tag{249}$$

and we define

$$\begin{aligned}\int \frac{d\Omega}{(4\pi)}\Bigg\{&\frac{2\left(p_4\cdot p_1\right)-\left(p_4\cdot p_2\right)+3m^2}{(p_1\cdot p_2)} \\ &+\frac{m^2\left(p_4\cdot p_2\right)+m^2\left(p_4\cdot p_1\right)+m^4}{(p_1\cdot p_2)^2}-2\Bigg\} = \frac{C\left(p_1,p_2,p_4\right)}{(p_1\cdot p_2)^2}.\end{aligned} \tag{250}$$

Finally integrating first in $d\cos\theta$ and then in $d\mid \vec{p_4}\mid$ (we can integrate in $d\mid \vec{p_4}\mid$ in the interval $[0,+\infty)$ eliminating δ since the function is regular in $d\mid \vec{p_4}\mid$ in this range) we have

$$\hat{\sigma}_{bg} = \frac{1}{4\left(p_1 \cdot p_2\right)} \frac{1}{\left(2\pi\right)^2} \sum_{\pm} \frac{\delta\left(\mid \vec{p_4} \mid - \mid \vec{p_4} \mid_0^{\pm}\right)}{\mid f'\left(\mid \vec{p_4} \mid_0^{\pm}\right) \mid} g^2 \left(\frac{4}{3}\right) \left(\frac{m}{v}\right)^2 \frac{1}{4E_3E_4}$$
$$(4\pi)\left[-4\frac{\mid \vec{p_1} \mid}{\mid \vec{p_2} \mid} + \left\{2A\left(p_1, p_2, p_4\right) + \frac{m^2 - \left(p_2 \cdot p_1\right)}{2 \mid \vec{p_2} \mid\mid \vec{p_4} \mid}\right\} \log\left(1 + \frac{2 \mid \vec{p_4} \mid}{E_4 - \mid \vec{p_4} \mid}\right)\right.$$
$$\left. - m^2 B\left(p_1, p_2, p_4\right) + \frac{m^2 - \left(p_2 \cdot p_1\right)}{\mid \vec{p_2} \mid^2} + \frac{C\left(p_1, p_2, p_4\right)}{\left(p_1 \cdot p_2\right)^2}\right] \mid \vec{p_4} \mid^2 d \mid \vec{p_4} \mid$$

$$= \frac{1}{4\left(p_1 \cdot p_2\right)} \frac{1}{\left(2\pi\right)^2} \sum_{\pm} \frac{1}{\mid f'\left(\mid \vec{p_4} \mid_0^{\pm}\right) \mid} g^2 \left(\frac{4}{3}\right) \left(\frac{m}{v}\right)^2 \frac{1}{4E_3E_4^{\pm}}$$
$$(4\pi)\left[-4\frac{\mid \vec{p_1} \mid}{\mid \vec{p_2} \mid} + \left\{2A\left(p_1, p_2, p_4^{\pm}\right) + \frac{m^2 - \left(p_2 \cdot p_1\right)}{2 \mid \vec{p_2} \mid\mid \vec{p_4} \mid_0^{\pm}}\right\} \log\left(1 + \frac{2 \mid \vec{p_4} \mid}{E_4^{\pm} - \mid \vec{p_4} \mid}\right)\right.$$
$$\left. - m^2 B\left(p_1, p_2, p_4^{\pm}\right) + \frac{m^2 - \left(p_2 \cdot p_1\right)}{\mid \vec{p_2} \mid^2} + \frac{C\left(p_1, p_2, p_4^{\pm}\right)}{\left(p_1 \cdot p_2\right)^2}\right] \mid \vec{p_4} \mid_0^{\pm 2}. \tag{251}$$

Again we notice a logarithmic term, which is finite thanks to the presence of the heavy quark mass (here denoted as the particle with momentum p_4), and which arises from the collinear emissions of heavy quarks.

Similarly to the results of the previous section, recognising in this result the analytical form of the correct coefficient, which should multiply the logarithmic terms, is difficult. We should express our result in terms of the kinematics variable $\hat{\tau}$ previously defined, and find the correct dependence from the splitting functions $P_{qg}(\tau)$ and $P_{qq}(\tau)$, as we have observed for the computations performed in dimensional regularization. The consistence of these computations with those presented in the previous section and done by authors of Ref. [9] have still to be checked numerically. However, the steps and the main features of the partonic cross sections computation are those described in these last subsections.

7 Outlook and conclusions

In this thesis we have discussed the generalization of the so-called FONLL method to hadronic processes, in order to account for the presence of initial-state heavy quarks also in the massive scheme. After revising the application of the standard FONLL method to deep inelastic scattering, and the way it has to be generalized in order to account for an intrinsic fitted charm, we have shown how the FONLL prescription can be used also to study heavy quarks in an hadronic process, specializing then the computations to the case of Higgs production in bottom quark fusion. Explicit results for this problem have then been reported. We have then discussed the general formalism which allows to consider heavy quarks in the initial state of an hadronic process also in the massive scheme, and subsequently we have derived explicit result for Higgs production in bottom quark fusion, finding the appropriate corrections which have to be added to the previous results.

Concerning Higgs production in bottom quark fusion, we have seen how, expanding the partonic cross section in power of the strong coupling, up to order α_s the analytical expressions we get are equivalent to those obtained in Ref. [9], apart from terms accounting for intrinsic heavy quark. The latter are a distinctive features of our approach, and they allow us to also consider a possible intrinsic heavy quark component. Another way to express these results is by noticing that the difference term, namely the difference between the massless scheme and the massless limit of the massive one, is identically zero up to order $O(\alpha_s)$, not only subleading as in the standard FONLL. Comparing these results with those for the analogue FONLL generalization in DIS, we therefore see that up to $O(\alpha_s)$ our discussion leads to the same conclusions: as for hadronic processes, the FONLL scheme with initial-state heavy quark in the massive scheme gives results obtained by a convolution between the massless PDFs and the massive partonic cross sections, and the difference term vanishes without expanding the heavy quark PDFs.

We have then described the guidelines to follow in order to obtain the corrections to standard FONLL up to order $O(\alpha_s^2)$, giving full analytical expression for them in terms of massive partonic cross sections and their massless limits, working out explicitly the proper matching condition between PDFs in massive and massless scheme up to order $O(\alpha_s^2)$. We also make some considerations about how this discussion could be extended at higher orders, following the analogue considerations made for DIS. We suggest that the difference term could indeed be identically zero also at higher order, but this possibility has to be checked further.

Finally, using explicit analytical result for partonic cross sections of Ref. [9], we obtain the full analytical expression for our result, presented in Eqs. (115), (119), (157), (165) and (169). In the final part of the work we have presented step by step detailed calculations of the partonic cross sections, regularizing the collinear divergences due to heavy quarks emissions with a finite heavy quark mass. These cross sections should then be compared with those of Ref. [9], used previously to get the final analytical expressions of our results.

In order to complete the work presented in this thesis, a first assessment of the phenomenological impact of a possible non-vanishing fitted heavy-quark should be performed. First we should verify that the modification to the FONLL scheme, expressed by Eq. (94), is indeed negligible in the absence of an intrinsic heavy quark, namely when, in our example, $f_b^{(5)}(x, m_b^2) = f_{\bar{b}}^{(5)}(x, m_b^2) = 0$, as already noticed before. Then we should consider a hadronic process in which the heavy quark involved does have an intrinsic component, and study the phenomenology, if necessary considering also some intrinsic heavy quarks models.

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