NNPDF studies with Heavy Quarks

Maria Ubiali, Juan Rojo

The **NNPDF Collaboration**: R. D. Ball, V. Bertone,
F. Cerutti, L. Del Debbio, S. Forte,
A. Guffanti, J. I. Latorre, J. R., M. U.

INFN, Sezione di Milano

PDF4LHC, QCD at the LHC workshop
ICT Trento 28/09/2010

Outline

What will not be discussed here

- Basics of the NNPDF methodology → M. U.'s talks
- Heavy quarks in DIS Theory and the FONLL GM scheme → P. Nason's talk

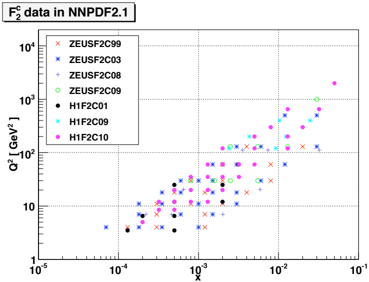
What we will talk about

- Impact of heavy quarks on PDFs and LHC observables
- Impact of the values of m_c and m_b
- PDFs in fixed-flavour number schemes

IMPACT OF HEAVY QUARK MASS EFFECTS IN THE NNPDF FRAMEWORK

The NNPDF2.1 analysis

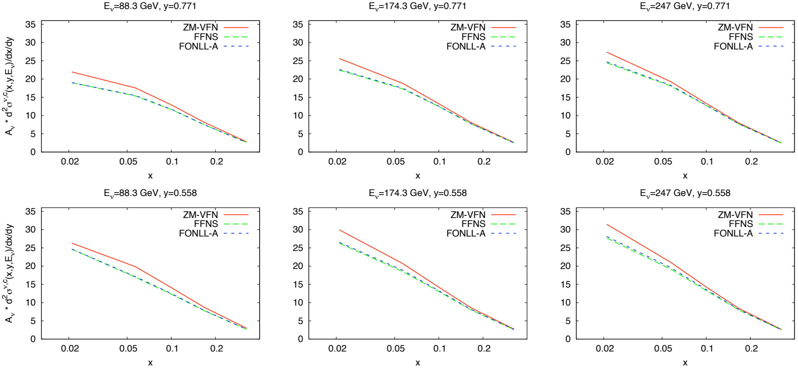
- FONLL-A-Damp as a General Mass scheme for NC and CC DIS observables
- Same dataset as NNPDF2.0 (arXiv:1002.4407), supplemented with HERA F_2^c data
- All results shown still preliminary



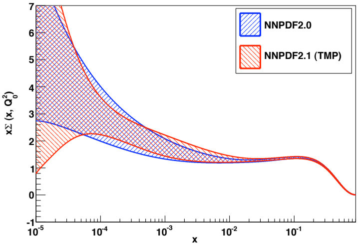
- For details on the FONLL GM scheme and its implementation in the NNPDF **FastKernel** framework, see J. Rojo's talks at PDF4LHC 01/10 and 07/10

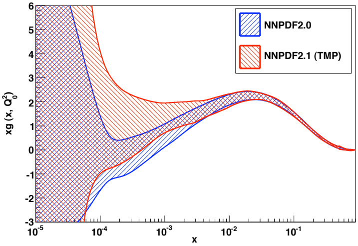
FONLL-A for Charged Current

The FONLL-A GM scheme also applies to CC structure functions
In the NuTeV kinematical region → FONLL-A very close to FFNS
(Les Houches heavy quark benchmark settings)



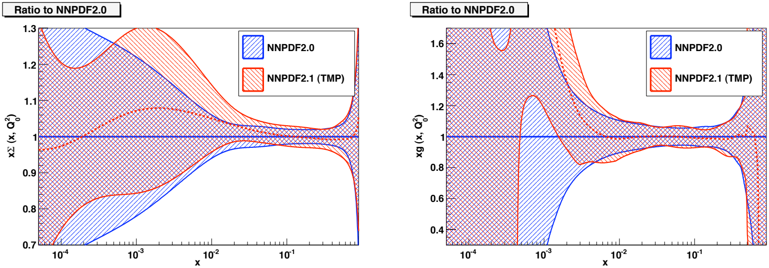
The NNPDF2.1 analysis - PDFs at $Q_0^2 = 2\ \mathrm{GeV}^2$





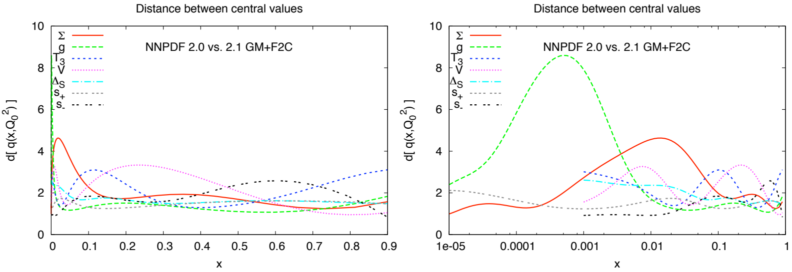
- HQ mass effects and F_2^c data enhance the singlet and the gluon PDFs at moderate and small-x
- NNPDF2.1 always within 1–sigma of NNPDF2.0 $\rightarrow$ HQ effects important though not dramatic
- Harder small-x gluon partly from constraints of $F_2^c(x, Q^2)$ data

The NNPDF2.1 analysis - PDFs at $Q_0^2 = 2$ GeV^2



- HQ mass effects and F_2^c data enhance the singlet and the gluon PDFs at moderate and small-x
- NNPDF2.1 always within 1–sigma of NNPDF2.0 $\rightarrow$ HQ effects important though not dramatic
- Harder small-x gluon partly from constraints of $F_2^c(x, Q^2)$ data

The NNPDF2.1 analysis - PDFs at $Q_0^2 = 2$ GeV^2



- Compute distances between PDF sets to quantify HQ impact

$$d^2\left(\langle q^{(1)}\rangle, \langle q^{(2)}\rangle\right) = \frac{\left(\langle q^{(1)}\rangle_{(1)} - \langle q^{(2)}\rangle_{(2)}\right)^2}{\sigma^2_{(1)}[\langle q^{(1)}\rangle] + \sigma^2_{(2)}[\langle q^{(2)}\rangle]} \; , \quad \sigma^2_{(i)}[\langle q^{(i)}\rangle] = \frac{1}{N^{(i)}_{\rm rep}}\sigma^2_{(i)}[q^{(i)}] \quad (1)$$

- $d \sim 5$ for the singlet at $x \sim 10^{-2}$ at $Q_0^2 = 2$ GeV^2
- $d \sim 8$ for the gluon at $x \sim 10^{-3}$ at $Q_0^2 = 2$ GeV^2

FONLL: treatment of subleading terms

- The FONLL F_{2c} structure function reads

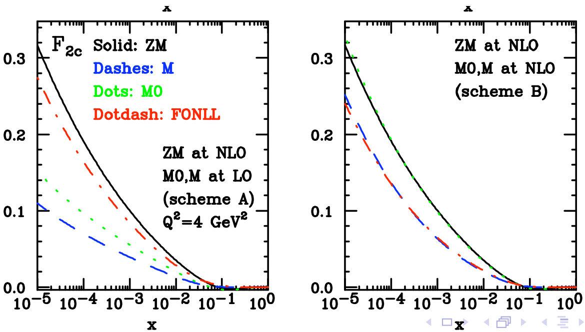
$$F_{2c}^{\mathrm{fonll}}(x,Q^2) = F_{2c}^{(n_l)}(x,Q^2)$$
$$+\Theta\left(Q^2-m_c^2\right)\left(1-\frac{m_c^2}{Q^2}\right)^2\left(F_{2c}^{(n_l+1)}(x,Q^2)-F_{2c}^{(n_l,0)}(x,Q^2)\right)$$

with $F_{2c}^{(n_l,0)}$ the massless limit of $F_{2c}^{(n_l)}$

- The *difference term* $\left(F_{2c}^{(n_l+1)}-F_{2c}^{(n_l,0)}\right)$ is $\mathcal{O}\left(\alpha_s^2\right)$ for $Q^2 \gtrsim m_c^2$, but numerically it turns out to be non-negligible
$\rightarrow$ can be suppressed by terms that go to 1 when $Q^2 \gg m_c^2$
- Possible choices are a threshold damping factor, or different forms of the χ–prescription
- This threshold ambiguity is an inherent theoretical uncertainty to any General–Mass scheme, can it be minimized?

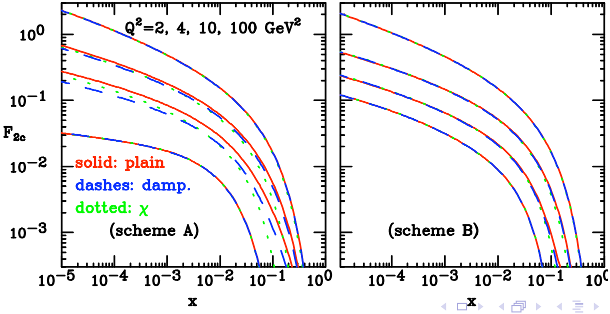
FONLL: treatment of subleading terms

- FONLL allows to combine different perturbative orders in ZM and FFNS terms
- FONLL-A combines the ZM scheme at $\mathcal{O}(\alpha_s)$ with the FFNS scheme at $\mathcal{O}(\alpha_s)$
  $\rightarrow$ Identical to S-ACOT
- FONLL-B combines the ZM scheme at $\mathcal{O}(\alpha_s)$ with the FFNS scheme at $\mathcal{O}(\alpha_s^2)$
- Advantage I: take into account consistently $\mathcal{O}(\alpha_s^2)$ massive contributions, phenomenologically important at small x and Q^2



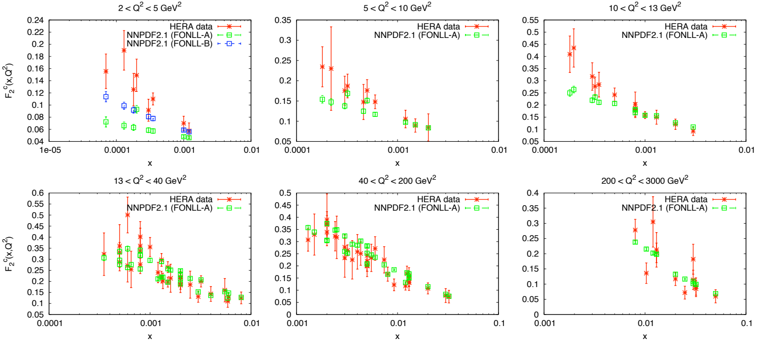
FONLL: treatment of subleading terms

- FONLL allows to combine different perturbative orders in ZM and FFNS terms
- FONLL-A combines the ZM scheme at $\mathcal{O}(\alpha_s)$ with the FFNS scheme at $\mathcal{O}(\alpha_s)$
  $\rightarrow$ Identical to S-ACOT
- FONLL-B combines the ZM scheme at $\mathcal{O}(\alpha_s)$ with the FFNS scheme at $\mathcal{O}(\alpha_s^2)$
- Advantage II: reduction of dependence on arbitrary threshold prescription – $F_{2c}^{\rm fonll} \sim F_{2c}^{(n_l)}$ up to moderate Q^2, with accuracy of $F_{2c}^{(n_l)}$ $\mathcal{O}(\alpha_s^2)$



Impact of F_2^c data in NNPDF2.1

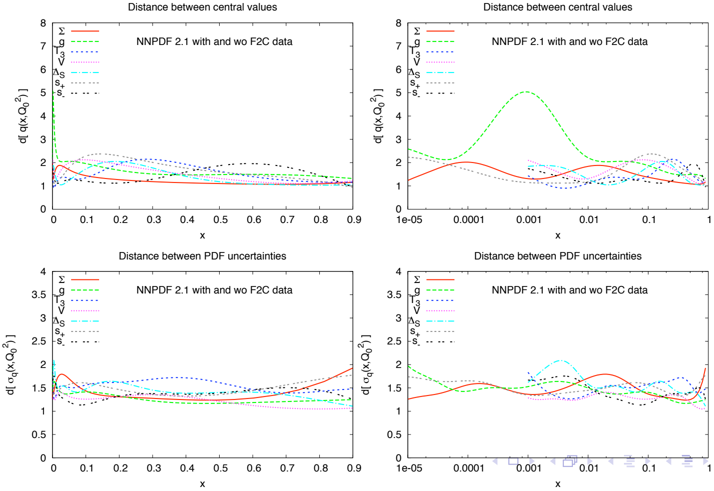
Good description of F_2^c data except at the smallest x and Q^2 bins
FONLL-A does not account for large $\mathcal{O}\left(\alpha_s^2\right)$ corrections to F_2^c in the FFNS



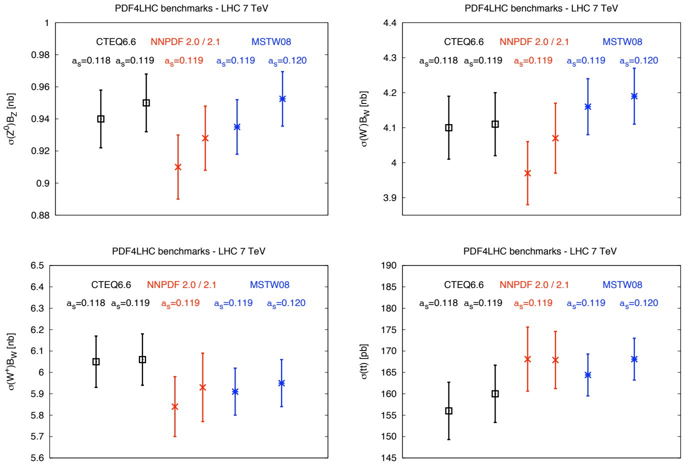
Update analysis with Combined HERA F_2^c dataset and with the FONLL-B GM scheme

Impact of F_2^c data in NNPDF2.1

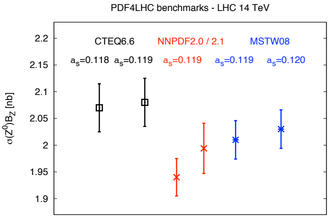
F_2^c data lead to an important constraint on the small-x gluon
$\rightarrow \sim 1/2$–sigma shift at $x \sim 10^{-3}$

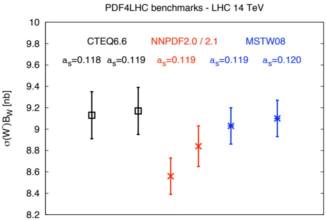


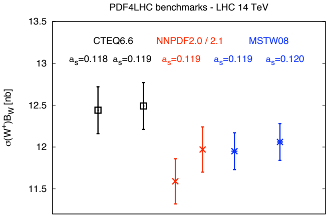
Impact on LHC observables - 7 and 14 TeV

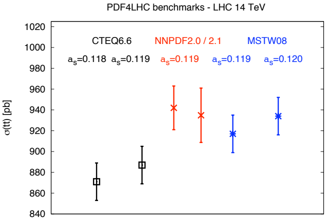


Impact on LHC observables - 7 and 14 TeV

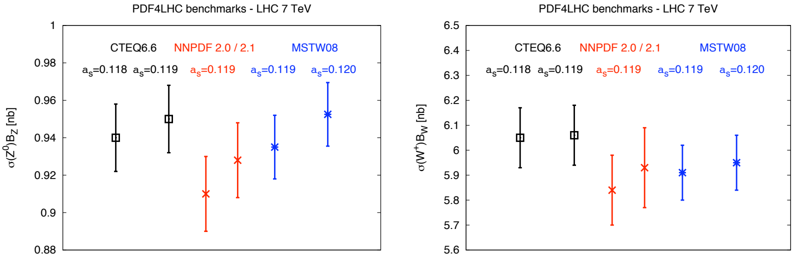








Impact on LHC observables - 7 and 14 TeV



- HQ mass effects and F_2^c data amount to an about $\sim$ 1–sigma shift in LHC observables at 7 TeV and at 14 TeV
- NNPDF2.1 predictions in excellent agreement with MSTW08 for all observables
- Only marginal agreement with CTEQ6.6 for most observables (also Higgs)
- Using common α_s increases the agreement

HEAVY QUARK PDFs

Heavy quark PDFs

- Heavy quark PDFs, $c(x, Q^2)$ and $b(x, Q^2)$, are much more dependent in the heavy quark mass chosen than on the VFN scheme adopted
- PDG values for HQ masses given by (NNLO scheme transformation)

$$m_c^{\bar{\mathrm{MS}}} = 1.27^{+0.07}_{-0.11}\ \mathrm{GeV} \to m_c^{\mathrm{pole}} \sim 1.73^{+0.07}_{-0.10}\ \mathrm{GeV}$$

$$m_b^{\bar{\mathrm{MS}}} = 4.20^{+0.17}_{-0.07}\ \mathrm{GeV} \to m_b^{\mathrm{pole}} \sim 4.90^{+0.18}_{-0.08}\ \mathrm{GeV}$$

  but perturbative expansion for $m_c^{\bar{\mathrm{MS}}}$ poorly convergent
- Crucial problem $\to$ Define best possible estimates of m_c and m_b and their associated uncertainties (analogously to the $\alpha_s(M_Z)$ case)
- The issue of the possibility of extracting m_c, m_b from the global fit should be separated from the choice of best m_h and δm_h, determined from many other external measurements

Impact of δm_c on LHC observables - 7 TeV

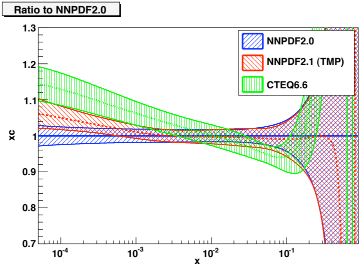
NNPDF2.1 analysis repeated for different m_c values

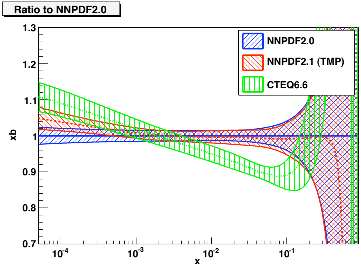
	$W^+ B_{l\nu}$	$W^- B_{l\nu}$	$Z^0 B_{l\bar{l}}$	$t\bar{t}$
$m_c = 1.41$	5.93 ± 0.16	4.07 ± 0.10	0.930 ± 0.02	167 ± 7
$m_c = 1.58$	6.04 ± 0.10	4.11 ± 0.07	0.945 ± 0.013	164 ± 5
$m_c = 1.70$	6.10 ± 0.15	4.16 ± 0.10	0.956 ± 0.02	163 ± 7

- Non-negligible impact of m_c variations, but not dramatic
- Uncertainties $\delta m_c \sim 0.10$ (PDG uncertainty) induce variations in $\sigma\left(W^{\pm}\right)$ and $\sigma\left(Z\right)$ below the 1–sigma PDF uncertainty
- Crucial problem $\rightarrow$ Define best estimate for δm_c!
- Similar studies performed by MSTW and HERAPDF
- The correlation between cross-sections and m_c can be easily computed in the NNPDF approach

Heavy quark PDFs

Ratio to NNPDF2.0 at $Q^2 = 10^4$ GeV^2



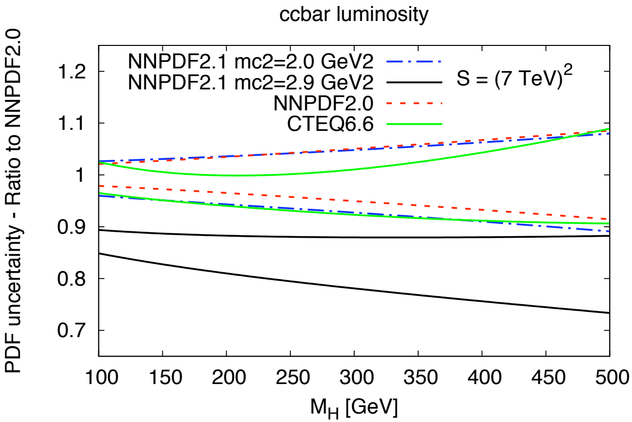


- Same pattern for $c(x, Q^2)$ and $b(x, Q^2)$ (Common evolution from singlet and gluon)
- Systematic discrepancy in b PDF for $x \in [0.01, 0.1]$ unrelated to ZM/GM differences, rather to different choices for m_b

Heavy quark PDFs

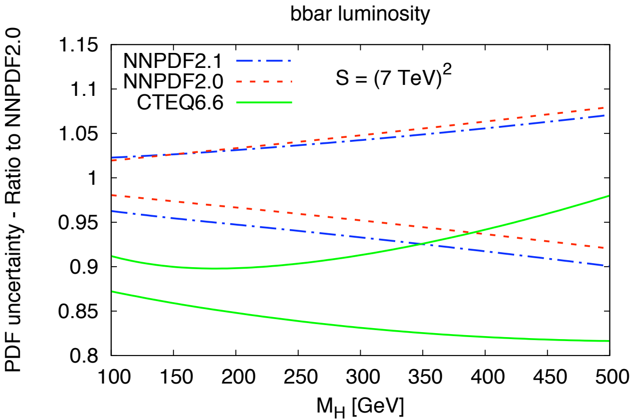
Luminosity $c\bar{c}$ at 7 TeV, Dependence on the charm quark mass and the GM scheme

The value of m_c more important than ZM/GM difference



The b PDF

The $b(x, Q^2)$ PDF is anticorrelated with m_b
Different values of m_b lead to very different $b\bar{b}$ luminosities
The differences in m_b much larger than the GM-ZM differences

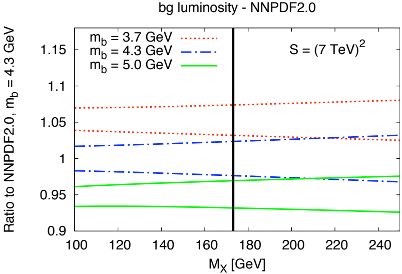


The b PDF

Taking into account uncertainty induced by m_b (correlated with the b–PDF) crucial for important LHC processes: single-top, MSSM Higgs, ...

Example: single top t-channel production: m_b–uncertainty $\gg$ PDF uncertainty

Differences both from PDF luminosity and from matrix element



NNPDF2.0	$\sigma(t)_{\rm t-channel}$
$m_b = 3.7$ GeV	46.77 ± 0.36 pb
$m_b = 4.3$ GeV	44.33 ± 0.32 pb
$m_b = 5.0$ GeV	41.04 ± 0.32 pb

The uncertainty in m_b and its correlation with the b PDF are crucial for b–initiated processes at the LHC

Crucial to determine best estimates for m_b, δm_b

PDF and heavy quark mass uncertainties

NNPDF2.1 sets for a range of different values of m_c and m_b will be provided
$\rightarrow$ Combined PDF+m_h uncertainties, exact error propagation to physical observables

$$\langle \mathcal{F} \rangle_{\rm rep} = \frac{1}{N_{\rm rep}} \sum_{i=1}^{N_{m_c}} \sum_{j=1}^{N_{m_b}} \sum_{k_{ij}=1}^{N_{\rm rep}^{(i,j)}} \mathcal{F}\left({\rm PDF}^{(k_{ij},i,j)}, m_c^{(i)}, m_b^{(j)} \right) \ ,$$

${\rm PDF}^{(k_{ij},i,j)}$ stands for the replica k_{ij} of the PDF fit obtained using $m_c^{(i)}$ and $m_b^{(j)}$

$$N_{\rm rep} = \sum_{i}^{N_{m_c}} \sum_{j}^{N_{m_b}} N_{\rm rep}^{(i,j)} \ ,$$

$N_{\rm rep}^{(i,j)}$ number of PDF replicas randomly selected from the fit obtained with $m_c^{(i)}$, $m_b^{(j)}$

$$N_{\rm rep}^{(i,j)} \propto \exp\left(- \frac{\left(m_c^{(i)} - m_c^{(0)} \right)^2}{2\delta_{m_c}^2} - \frac{\left(m_b^{(j)} - m_b^{(0)} \right)^2}{2\delta_{m_b}^2} \right) \ .$$

Important advantage: No extra CPU time required! (Set $N_{\rm rep} = 100$)

Another advantage: both $m_c^{(0)}$, $m_b^{(0)}$ and $\delta_{m_c}, \delta_{m_b}$ can be decided by the PDF user

PDFs WITH FIXED-FLAVOR NUMBER

PDFs with Fixed Flavor Number

- PDF in the Fixed Flavour $N_f = 3$ and $N_f = 4$ schemes important for LHC phenomenology
- FFN sets can easily be obtained from $N_f = 5$ GM PDF sets by matching PDFs and α_s at the HQ mass threshold

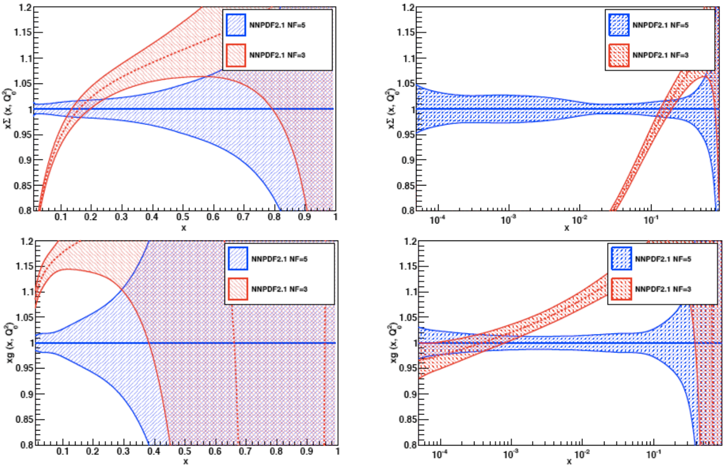
$$\mathrm{PDFs}^{(N_f)}(Q^2 = m_h^2) = \mathrm{PDFs}^{(N_f+1)}(Q^2 = m_h^2)$$

$$\alpha_s^{(N_f)}(Q^2 = m_h^2) = \alpha_s^{(N_f+1)}(Q^2 = m_h^2) \ ,$$

  and then evolving upwards with fixed N_f
- Bypass problems related to unknown massive FFN coefficient functions for jets and DY
- Same approach adopted by CT and MSTW

NNPDF2.1 $N_f = 3$ PDFs

$N_f = 3$ and $N_f = 4$ sets of NNPDF2.1 will be provided
Compare $N_f = 3$ with $N_f = 5$ PDFs at LHC scale $Q^2 = 10^4$ GeV^2



Summary

- The NNPDF2.1 analysis is based in the FONLL General Mass scheme for heavy quark effects. Will be released in the coming weeks.
- NNPDF2.1 sets with different values of m_c and m_b will be provided
- $N_f = 3$ and $N_f = 4$ PDF sets will also be provided
- The impact of m_c variations on LHC observables is comparable in size to GM/ZM differences
- The b–PDF depends crucial on the value of m_b $\rightarrow$ Important phenomenological impact in b–initiated LHC processes
- Within NNPDF, easy to compute and propagate the correlation between PDFs and heavy quark masses
- The choice of the heavy quark mass m_h can be as important as the ZM/GM difference $\rightarrow$ Crucial problem to converge on a common choice of *best estimates* for m_h and δm_h

Thanks for your attention!

Summary

- The NNPDF2.1 analysis is based in the FONLL General Mass scheme for heavy quark effects. Will be released in the coming weeks.
- NNPDF2.1 sets with different values of m_c and m_b will be provided
- $N_f = 3$ and $N_f = 4$ PDF sets will also be provided
- The impact of m_c variations on LHC observables is comparable in size to GM/ZM differences
- The b–PDF depends crucial on the value of m_b $\to$ Important phenomenological impact in b–initiated LHC processes
- Within NNPDF, easy to compute and propagate the correlation between PDFs and heavy quark masses
- The choice of the heavy quark mass m_h can be as important as the ZM/GM difference $\to$ Crucial problem to converge on a common choice of *best estimates* for m_h and δm_h

Thanks for your attention!

Summary

- The NNPDF2.1 analysis is based in the FONLL General Mass scheme for heavy quark effects. Will be released in the coming weeks.
- NNPDF2.1 sets with different values of m_c and m_b will be provided
- $N_f = 3$ and $N_f = 4$ PDF sets will also be provided
- The impact of m_c variations on LHC observables is comparable in size to GM/ZM differences
- The b–PDF depends crucial on the value of m_b $\rightarrow$ Important phenomenological impact in b–initiated LHC processes
- Within NNPDF, easy to compute and propagate the correlation between PDFs and heavy quark masses
- The choice of the heavy quark mass m_h can be as important as the ZM/GM difference $\rightarrow$ Crucial problem to converge on a common choice of *best estimates* for m_h and δm_h

Thanks for your attention!

Summary

- The NNPDF2.1 analysis is based in the FONLL General Mass scheme for heavy quark effects. Will be released in the coming weeks.
- NNPDF2.1 sets with different values of m_c and m_b will be provided
- $N_f = 3$ and $N_f = 4$ PDF sets will also be provided
- The impact of m_c variations on LHC observables is comparable in size to GM/ZM differences
- The b–PDF depends crucial on the value of m_b $\rightarrow$ Important phenomenological impact in b–initiated LHC processes
- Within NNPDF, easy to compute and propagate the correlation between PDFs and heavy quark masses
- The choice of the heavy quark mass m_h can be as important as the ZM/GM difference $\rightarrow$ Crucial problem to converge on a common choice of *best estimates* for m_h and δm_h

Thanks for your attention!

Summary

- The NNPDF2.1 analysis is based in the FONLL General Mass scheme for heavy quark effects. Will be released in the coming weeks.
- NNPDF2.1 sets with different values of m_c and m_b will be provided
- $N_f = 3$ and $N_f = 4$ PDF sets will also be provided
- The impact of m_c variations on LHC observables is comparable in size to GM/ZM differences
- The b–PDF depends crucial on the value of $m_b \rightarrow$ Important phenomenological impact in b–initiated LHC processes
- Within NNPDF, easy to compute and propagate the correlation between PDFs and heavy quark masses
- The choice of the heavy quark mass m_h can be as important as the ZM/GM difference $\rightarrow$ Crucial problem to converge on a common choice of *best estimates* for m_h and δm_h

Thanks for your attention!

Summary

- The NNPDF2.1 analysis is based in the FONLL General Mass scheme for heavy quark effects. Will be released in the coming weeks.
- NNPDF2.1 sets with different values of m_c and m_b will be provided
- $N_f = 3$ and $N_f = 4$ PDF sets will also be provided
- The impact of m_c variations on LHC observables is comparable in size to GM/ZM differences
- The b–PDF depends crucial on the value of m_b $\rightarrow$ Important phenomenological impact in b–initiated LHC processes
- Within NNPDF, easy to compute and propagate the correlation between PDFs and heavy quark masses
- The choice of the heavy quark mass m_h can be as important as the ZM/GM difference $\rightarrow$ Crucial problem to converge on a common choice of *best estimates* for m_h and δm_h

Thanks for your attention!

Summary

- The NNPDF2.1 analysis is based in the FONLL General Mass scheme for heavy quark effects. Will be released in the coming weeks.
- NNPDF2.1 sets with different values of m_c and m_b will be provided
- $N_f = 3$ and $N_f = 4$ PDF sets will also be provided
- The impact of m_c variations on LHC observables is comparable in size to GM/ZM differences
- The b–PDF depends crucial on the value of m_b $\rightarrow$ Important phenomenological impact in b–initiated LHC processes
- Within NNPDF, easy to compute and propagate the correlation between PDFs and heavy quark masses
- The choice of the heavy quark mass m_h can be as important as the ZM/GM difference $\rightarrow$ Crucial problem to converge on a common choice of *best estimates* for m_h and δm_h

Thanks for your attention!

Summary

- The NNPDF2.1 analysis is based in the FONLL General Mass scheme for heavy quark effects. Will be released in the coming weeks.
- NNPDF2.1 sets with different values of m_c and m_b will be provided
- $N_f = 3$ and $N_f = 4$ PDF sets will also be provided
- The impact of m_c variations on LHC observables is comparable in size to GM/ZM differences
- The b–PDF depends crucial on the value of m_b $\rightarrow$ Important phenomenological impact in b–initiated LHC processes
- Within NNPDF, easy to compute and propagate the correlation between PDFs and heavy quark masses
- The choice of the heavy quark mass m_h can be as important as the ZM/GM difference $\rightarrow$ Crucial problem to converge on a common choice of *best estimates* for m_h and δm_h

Thanks for your attention!

Summary

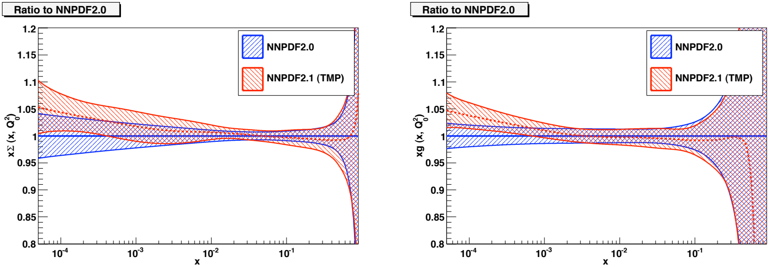
- The NNPDF2.1 analysis is based in the FONLL General Mass scheme for heavy quark effects. Will be released in the coming weeks.
- NNPDF2.1 sets with different values of m_c and m_b will be provided
- $N_f = 3$ and $N_f = 4$ PDF sets will also be provided
- The impact of m_c variations on LHC observables is comparable in size to GM/ZM differences
- The b–PDF depends crucial on the value of m_b $\rightarrow$ Important phenomenological impact in b–initiated LHC processes
- Within NNPDF, easy to compute and propagate the correlation between PDFs and heavy quark masses
- The choice of the heavy quark mass m_h can be as important as the ZM/GM difference $\rightarrow$ Crucial problem to converge on a common choice of *best estimates* for m_h and δm_h

Thanks for your attention!

EXTRA MATERIAL

The NNPDF2.1 analysis - PDFs at $Q_0^2 = 10^4$ GeV^2

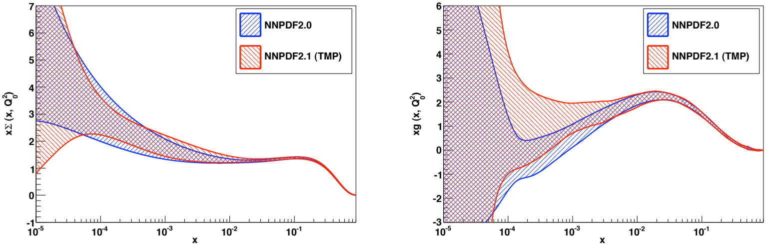
Compare PDFs at the LHC scale → Assess effects of quark-gluon mixing in DGLAP evolution



Note greatly reduced small-x PDF uncertainties

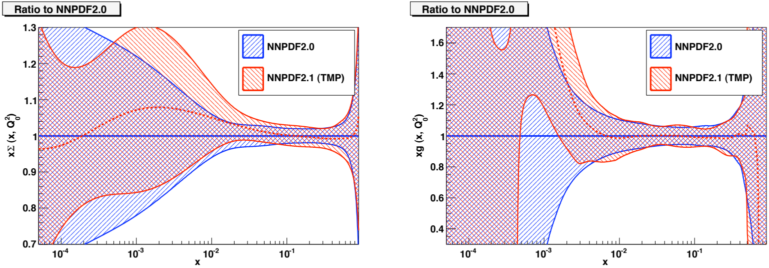
NNPDF2.0 and 2.1 always consistent within uncertainties

The NNPDF2.1 analysis - PDFs at $Q_0^2 = 2\ \mathrm{GeV}^2$



- HQ mass effects and F_2^c data enhance the singlet and the gluon PDFs at moderate and small-x
- NNPDF2.1 always within 1–sigma of NNPDF2.0 → HQ effects important though not dramatic
- Harder small-x gluon partly from constraints of $F_2^c(x, Q^2)$ data

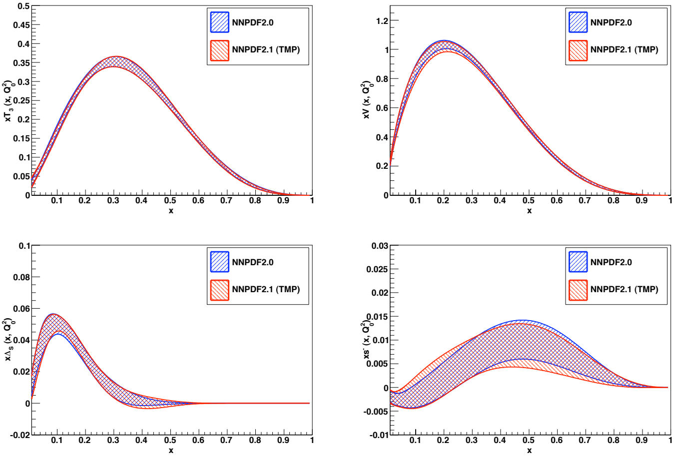
The NNPDF2.1 analysis - PDFs at $Q_0^2 = 2\ \mathrm{GeV}^2$



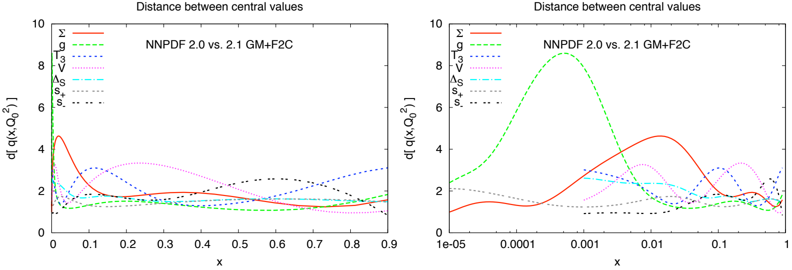
- HQ mass effects and F_2^c data enhance the singlet and the gluon PDFs at moderate and small-x
- NNPDF2.1 always within 1–sigma of NNPDF2.0 $\rightarrow$ HQ effects important though not dramatic
- Harder small-x gluon partly from constraints of $F_2^c(x, Q^2)$ data

The NNPDF2.1 analysis - PDFs at $Q_0^2 = 2$ GeV^2

Large-x valence PDFs consistently unaffected by HQ effects



The NNPDF2.1 analysis - PDFs at $Q_0^2 = 2\ \mathrm{GeV}^2$



- Compute distances between PDF sets to quantify HQ impact

$$d^2\left(\langle q^{(1)}\rangle, \langle q^{(2)}\rangle\right) = \frac{\left(\langle q^{(1)}\rangle_{(1)} - \langle q^{(2)}\rangle_{(2)}\right)^2}{\sigma^2_{(1)}[\langle q^{(1)}\rangle] + \sigma^2_{(2)}[\langle q^{(2)}\rangle]}\ , \quad \sigma^2_{(i)}[\langle q^{(i)}\rangle] = \frac{1}{N^{(i)}_{\rm rep}}\sigma^2_{(i)}[q^{(i)}] \tag{2}$$

- $d \sim 5$ for the singlet at $x \sim 10^{-2}$ at $Q_0^2 = 2\ \mathrm{GeV}^2$
- $d \sim 8$ for the gluon at $x \sim 10^{-3}$ at $Q_0^2 = 2\ \mathrm{GeV}^2$

The NNPDF2.1 analysis - Dataset description (Prel.)

Experiment/Set	2.0	2.0 + GM	2.1
Total	**1.20**	**1.23**	**1.21**
NMC-pd	1.04	*1.03*	0.96
NMC	1.69	*1.51*	1.61
SLAC	1.30	*1.31*	1.31
BCDMS	1.30	*1.19*	1.21
HERA1-av	1.13	*1.28*	1.11
HERA1-NCep	**1.32**	*1.56*	**1.29**
HERA1-NCem	0.85	*0.88*	0.82
HERA1-CCep	0.97	*0.97*	0.96
HERA1-CCem	0.57	*0.57*	0.57
ZEUS-H2	1.24	*1.27*	1.23
ZEUSF2C	*1.80*	2.14	1.89
H1F2C	*1.67*	1.70	1.59
CHORUS	1.20	*1.18*	1.19
NTVDMN	0.70	*0.71*	0.71

- Overall fit quality almost identical between GM and ZM fits
- Quality of FT DIS data (NMC, BCDMS) improves in the GM fit as compared to ZM
- Quality of fit to HERA-I data unaffected
- Heavy quark effects are absorbed into the PDFs in the ZM fit of HERA1-NCep data
- Fit to F_2^c data not completely satisfactory (see after)

FONLL in a nutshell

- Express the massive result $F^{(n_l)}$ in terms of the massless PDFs and α_s (non trivial from $\mathcal{O}\left(\alpha_s^2\right)$)

$$F^{(n_l)}(x,Q^2) = x\int_x^1 \frac{dy}{y} \sum_{i=q,\bar{q},g} B_i\left(\frac{x}{y},\frac{Q^2}{m^2},\alpha_s^{(n_l+1)}(Q^2)\right) f_i^{(n_l+1)}(y,Q^2),$$

- Define massless limit of the massive computation as

$$F^{(n_l,0)}(x,Q^2) \equiv x\int_x^1 \frac{dy}{y} \sum_{i=q,\bar{q},g} B_i^{(0)}\left(\frac{x}{y},\frac{Q^2}{m^2},\alpha_s^{(n_l+1)}(Q^2)\right) f_i^{(n_l+1)}(y,Q^2),$$

$$\lim_{m\to 0}\left[B_i\left(x,\frac{Q^2}{m^2}\right) - B_i^{(0)}\left(x,\frac{Q^2}{m^2}\right)\right] = 0$$

- The FONLL approximation is then

$$F^{\rm FONLL}(x,Q^2) \equiv F^{(d)}(x,Q^2) + F^{(n_l)}(x,Q^2),$$
$$F^{(d)}(x,Q^2) \equiv \left[F^{(n_l+1)}(x,Q^2) - F^{(n_l,0)}(x,Q^2)\right]$$

Important technical advantage: PDFs and α_s expressed always in the (n_l+1) scheme

FONLL in a nutshell

- Far from threshold, $Q^2 \gg m^2$ $F^{(n_l,0)}(x,Q^2) \sim F^{(n_l)}(x,Q^2) \rightarrow$ the massless computation recovered

$$F^{\text{FONLL}}(x,Q^2) \sim F^{(n_l+1)}(x,Q^2)$$

- Near threshold the "difference term" is formally higher order but unreliable, so one can correct it by mass suppressed terms, using for example a damping factor (FONLL default)

$$F^{(d,\,th)}(x,Q^2) \equiv f_{\rm thr}(x,Q^2)F^{(d)}(x,Q^2), \quad f_{\rm thr}(x,Q^2) = \Theta(Q^2-m^2)\left(1-\frac{Q^2}{m^2}\right)^2,$$

or some form of $\chi-$scaling,

$$F^{(d,\,\chi)}(x,Q^2) \equiv F^{(d)}(x,Q^2) = x\int_{\chi(x,Q^2)} \frac{dy}{y} C\left(\frac{\chi(x,Q^2)}{y},\alpha(Q^2)\right) f(y,Q^2),$$

$$F^{(d,\,\chi,v2)}(x,Q^2) \equiv F^{(d)}(\chi(x,Q^2),Q^2), \quad \chi = x\left(1+\frac{4m^2}{Q^2}\right).$$

The choice of threshold prescription represent an intrinsic ambiguity of the matching procedure. Can this ambiguity be minimized?

Perturbative ordering in FONLL

Three FONLL schemes for different ordering of the perturbative expansion can be defined:

1. Scheme A → $\mathcal{O}(\alpha_s)$ in massless and in massive
2. Scheme B → $\mathcal{O}(\alpha_s)$ in massless and $\mathcal{O}(\alpha_s^2)$ in massive
3. Scheme C → $\mathcal{O}(\alpha_s^2)$ in massless and in massive

In any of the three schemes, any threshold prescription can be implemented
These schemes can be related to existing approaches

1. Scheme A is identical to S-ACOT
2. Scheme B was formulated with similar scope as TR (use the information from the $\mathcal{O}(\alpha_s^2)$ massive computation in a NLO GM-VFN scheme), but they turn to be different
3. Scheme C should be S-ACOT at NNLO?