PDFs with theoretical uncertainties

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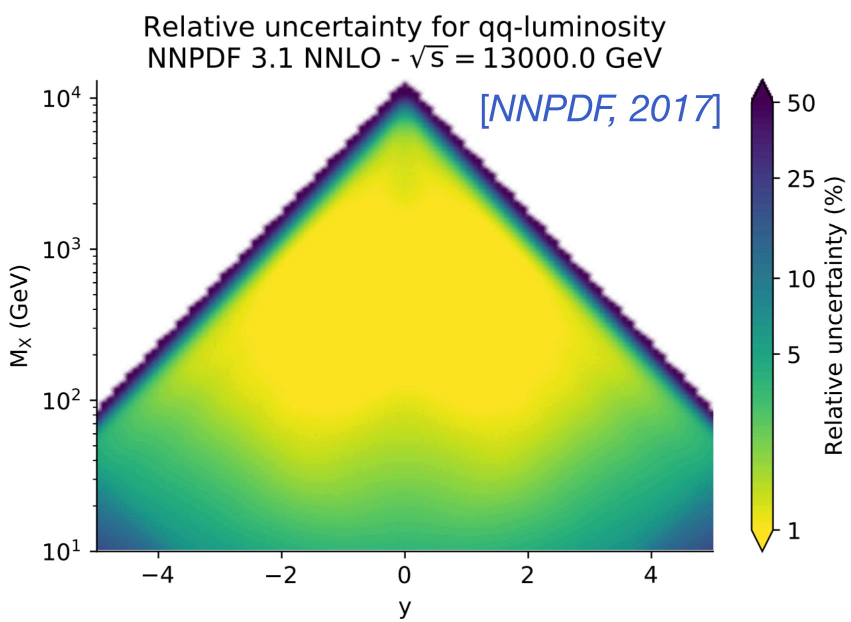
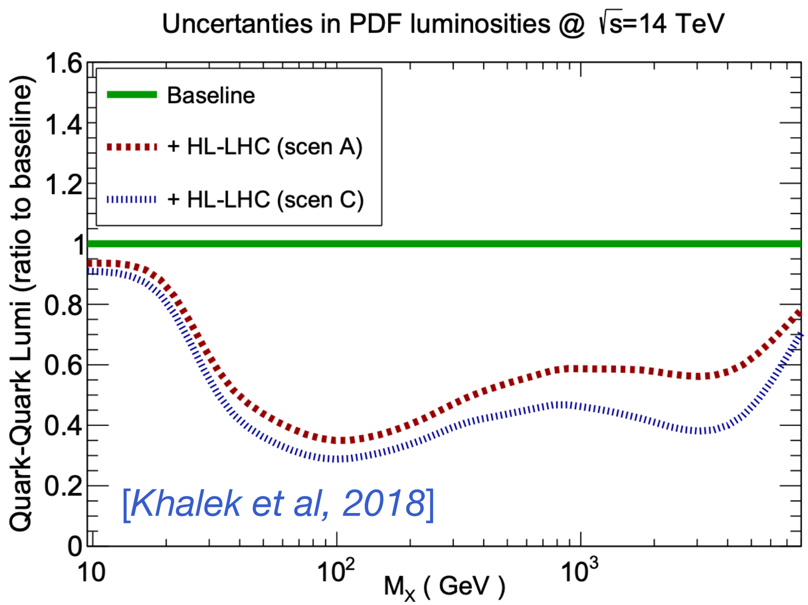
Ultimate Precision at Hadron Colliders
26 November 2019

State-of-the-art PDFs



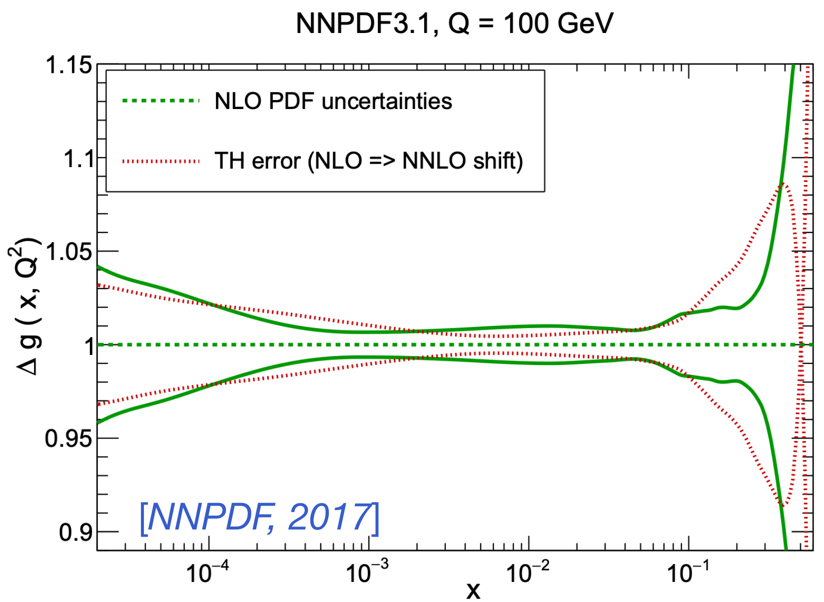


- PDFs now **high precision**: ~1% uncertainty in data region
- Uncertainties will get **smaller** with HL-LHC
- PDFs are **precise**, but are they **accurate**?

Theoretical uncertainties & PDFs

- Standard PDF fits use **fixed-order** partonic cross sections and **fixed-order** PDF evolution (**NNLO** for state-of-the-art PDFs)

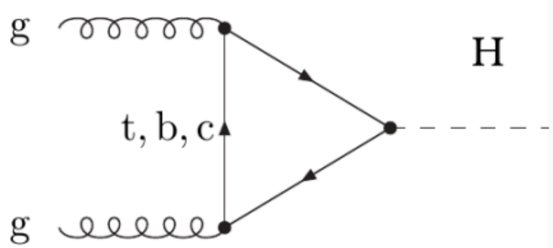
What is the **potential impact** of theoretical uncertainties in PDF fits?



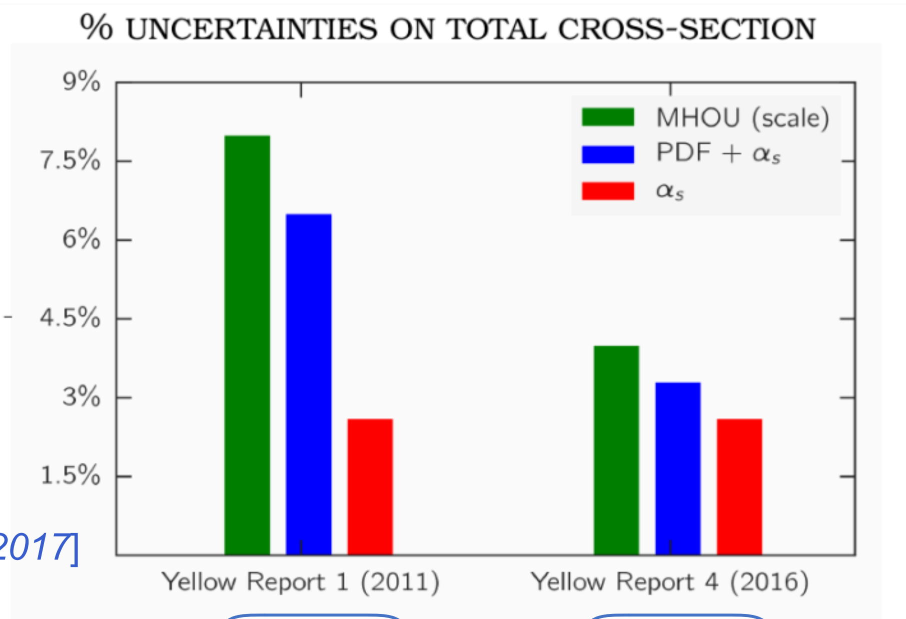
- **NNLO-NLO PDF shift** now of **same order** or **larger** than PDF uncertainties
- Should we worry about **accuracy** of PDFs? Looking forward: yes

Theoretical uncertainties at the LHC

Example: gluon-gluon fusion



[S. Forte, Lattice 2017]



NNLO

N^3LO

- **Missing higher-order uncertainties** (MHOUs) often dominant at LHC
- MHOUs are uncertainties due to **truncation** of series used in calculations, namely in **partonic cross sections** and **PDF evolution** (DGLAP equations)

Estimating MHOUs

Standard technique: **scale variations**

- Thinking behind method:

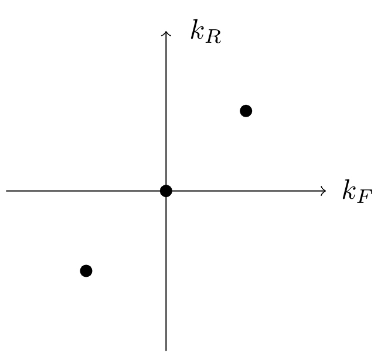
  1. μ_R, μ_F are **"unphysical"** scales that all-orders prediction cannot depend on
  2. Varying μ_R, μ_F in $O(\alpha_s^n)$ calculation generates $O(\alpha_s^{n+1})$ terms

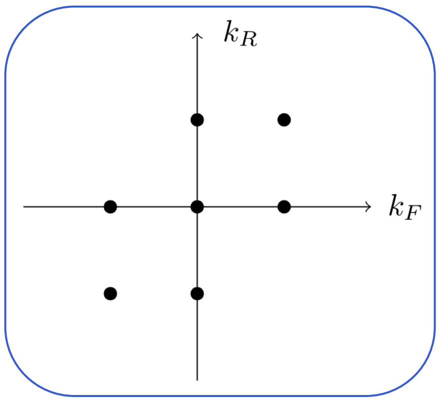
- Convention (for hadronic processes): vary μ_R in **partonic cross section** and μ_F in **PDF**, where

$$k_R, k_F \in \left(\frac{1}{2}, 1, 2\right)$$

$$k = \frac{\mu}{\mu_0}$$

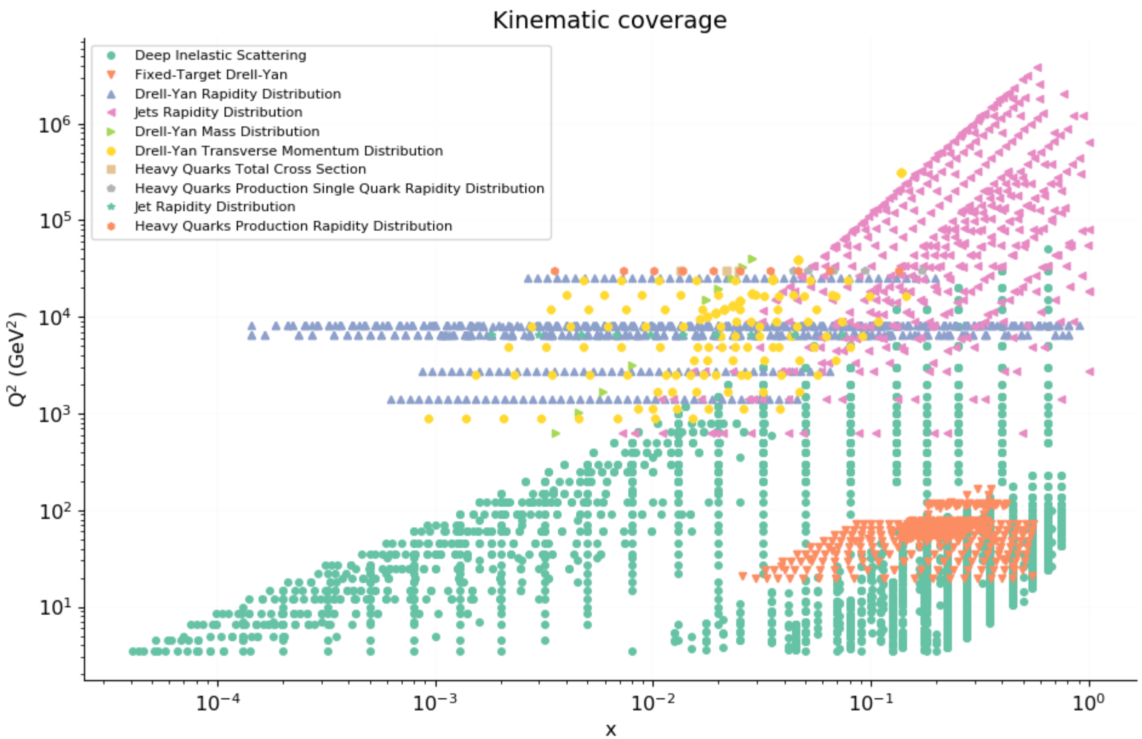
- Compute observable for different scale combinations and take **envelope**





HXSWG recommendation

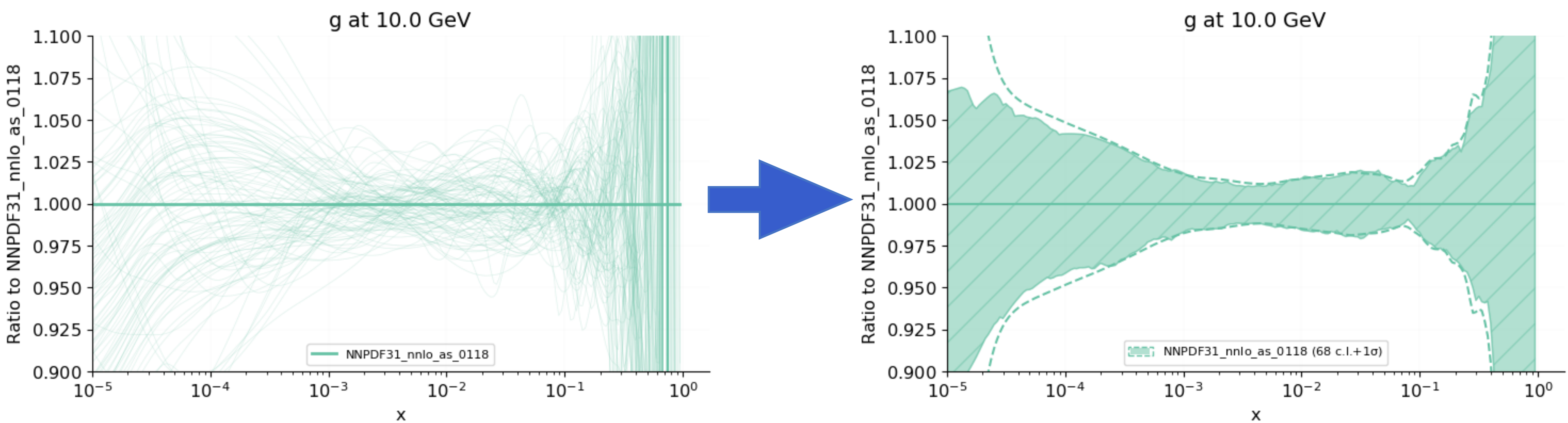
PDF determinations



How to **extend** scale variation to **global PDF fits**?

- **O(4000)** data points from **different processes**
- How to **correlate**? Common DGLAP evolution, different α_s dependence in partonic cross sections

Propagation of uncertainties in NNPDF



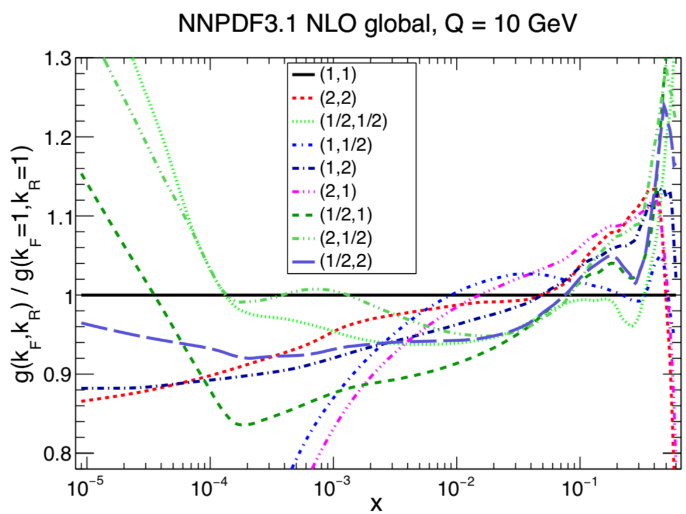
- Sample Monte Carlo replicas from distribution of data
- Fit PDFs for each data replica by minimising χ^2 $\rightarrow$ ensemble of PDF replicas

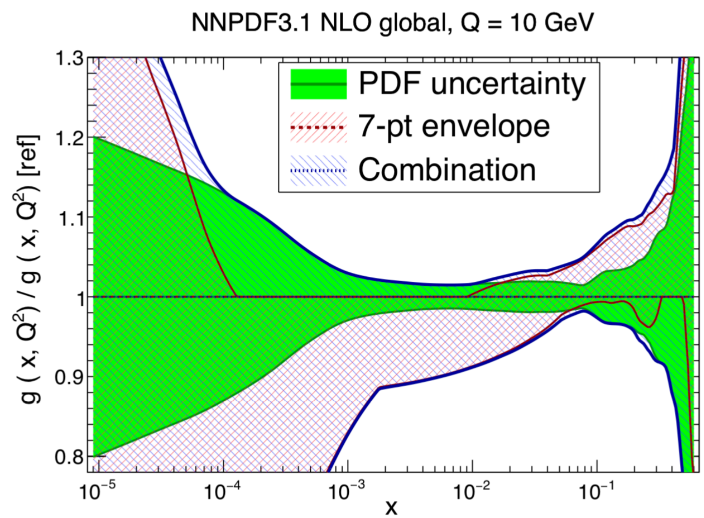
- Currently (e.g. NNPDF 3.1): **all replicas computed with central scales** and theoretical uncertainties **not included elsewhere** (e.g. in χ^2)

PDF fits with varied scales

Starting point for estimating MHOUs:

- Produce PDF fits for **range of scale combinations**
- Define MHOUs band as **envelope of central values**





- **Neglects correlations** in scale variations
- MHOUs only **estimated**, not **included** in PDF uncertainties

How to include **MHOUs** and their **correlations** in PDFs by accounting for them in the **fitting methodology**?

Approach I:
The theoretical covariance matrix

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https://doi.org/10.1140/epjc/s10052-019-7364-5

THE EUROPEAN PHYSICAL JOURNAL C

Check for updates

Letter

A first determination of parton distributions with theoretical uncertainties

Rabah Abdul Khalek[1,2], Richard D. Ball[3], Stefano Carrazza[4], Stefano Forte[4,a], Tommaso Giani[3], Zahari Kassabov[5], Emanuele R. Nocera[2], Rosalyn L. Pearson[3], Juan Rojo[1,2], Luca Rottoli[6,7], Maria Ubiali[8], Cameron Voisey[5], Michael Wilson[3]

Summary: *Eur. Phys. J. C (2019) 79: 838*
More details: *Eur. Phys. J. C (2019) 79: 931*

The theoretical covariance matrix

Experimental uncertainties propagated to PDFs via minimisation of **figure of merit**:

$$\chi^2 = (\text{data} - \text{theory})^T(\text{cov}_{\text{exp}})^{-1}(\text{data} - \text{theory})$$

Modify this to account for theory errors: *[R. D. Ball & A. Deshpande, 2018]*

$$\chi^2_{\text{tot}} = (\text{data} - \text{theory})^T(\text{cov}_{\text{exp}} + \text{cov}_{\text{th}})^{-1}(\text{data} - \text{theory})$$

Assumptions:

1. Theoretical uncertainties **independent** from experimental uncertainties
   → we are adding exp. and th. uncertainties in quadrature
2. Theoretical uncertainties are **Gaussianly distributed**

Applicable to other types of theoretical uncertainty, e.g. Monte Carlo, nuclear uncertainties *[R. D. Ball et al, 2018]*, …

A theoretical covariance matrix for MHOUs

Construct cov_{th} from **scale variations** to estimate:

1. MHOU on each point
2. Correlations between points

i, j: data points

k: scale combinations

$$\text{cov}_{\text{th,ij}} = \frac{1}{N}\sum_k \Delta_i^{(k)}\Delta_j^{(k)} \qquad \Delta_i^{(k)} = t_i(\mu_R, \mu_F) - t_i(\mu_{R,0}, \mu_{F,0})$$

Choices:

0. Definition of covariance matrix
1. Range of scale variation
2. Number of scale combinations (3, 7, …)
3. Correlation between scales (same process, different processes)
4. Process categorisation
5. Type of scale variation

A theoretical covariance matrix for MHOUs

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Choices:

0. Definition of covariance matrix
1. Range of scale variation → $\frac{1}{2} \leq k_F, k_R \leq 2$
2. Number of scale combinations (3, 7, …)
3. Correlation between scales (same process, different processes)
4. Process categorisation
5. Type of scale variation

A theoretical covariance matrix for MHOUs

Construct cov_{th} from **scale variations** to estimate:

1. MHOU on each point
2. Correlations between points

i, *j*: data points

k: scale combinations

$$\text{cov}_{\text{th,ij}} = \frac{1}{N} \sum_k \Delta_i^{(k)} \Delta_j^{(k)} \qquad \Delta_i^{(k)} = t_i(\mu_R, \mu_F) - t_i(\mu_{R,0}, \mu_{F,0})$$

Choices:

0. Definition of covariance matrix
1. Range of scale variation
2. Number of scale combinations (3, 7, …)
3. Correlation between scales (same process, different processes)
4. Process categorisation
5. Type of scale variation

How do we correlate scales in this multi-scale problem?

See next slides

A theoretical covariance matrix for MHOUs

Construct cov_{th} from **scale variations** to estimate:

1. MHOU on each point
2. Correlations between points

i, j: data points

k: scale combinations

$$\text{cov}_{\text{th,ij}} = \frac{1}{N} \sum_k \Delta_i^{(k)} \Delta_j^{(k)} \qquad \Delta_i^{(k)} = t_i(\mu_R, \mu_F) - t_i(\mu_{R,0}, \mu_{F,0})$$

Choices:

0. Definition of covariance matrix
1. Range of scale variation
2. Number of scale combinations (3, 7, …)
3. Correlation between scales (same process, different processes)
4. Process categorisation →
5. Type of scale variation

DIS neutral current

DIS charged current

Drell-Yan

Jets

Top

A theoretical covariance matrix for MHOUs

Construct cov_{th} from **scale variations** to estimate:

1. MHOU on each point
2. Correlations between points

i, j: data points

k: scale combinations

$$\mathrm{cov}_{\mathrm{th},ij} = \frac{1}{N}\sum_k \Delta_i^{(k)}\Delta_j^{(k)} \qquad \Delta_i^{(k)} = t_i(\mu_R, \mu_F) - t_i(\mu_{R,0}, \mu_{F,0})$$

Choices:

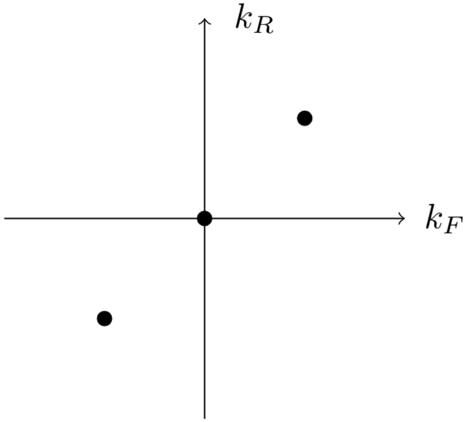
0. Definition of covariance matrix
1. Range of scale variation
2. Number of scale combinations (3, 7, …)
3. Correlation between scales (same process, different processes)
4. Process categorisation
5. Type of scale variation →

- Vary μ_R in $\hat{\sigma}$
- Vary μ_F in PDF (scale at which PDF is evaluated)

Example: 3-pt theoretical covariance matrix

i, j from ***same process***

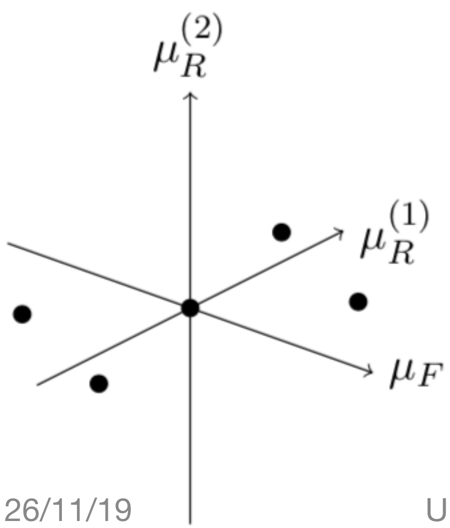
Assumptions: one μ_F in total, one μ_R per process



$$\mathrm{cov}_{\mathrm{th,ij}} = \frac{1}{2}\left\{\Delta_i(+,+)\Delta_j(+,+) + \Delta_i(-,-)\Delta_j(-,-)\right\}$$

i, j from ***different processes***

$$\mathrm{cov}_{\mathrm{th,ij}} = \frac{1}{4}\left\{(\Delta_i(+,+) + \Delta_i(-,-))(\Delta_j(+,+) + \Delta_j(-,-))\right\}$$



where

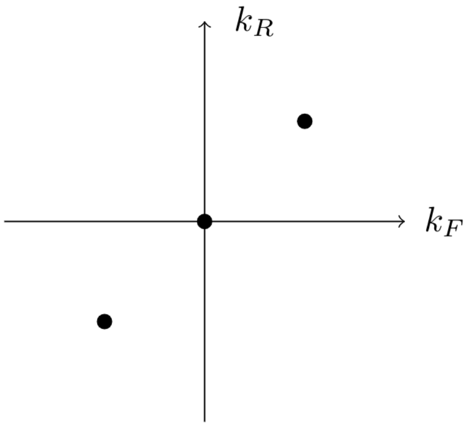
$$\Delta_i(+,+) = t_i(k_F = 2,\ k_R = 2) - t_i(k_F = 1,\ k_R = 1)$$

$$\Delta_i(-,-) = t_i\left(k_F = \frac{1}{2},\ k_R = \frac{1}{2}\right) - t_i(k_F = 1,\ k_R = 1)$$

Example: 3-pt theoretical covariance matrix

Assumptions: one μ_F in total, one μ_R per process

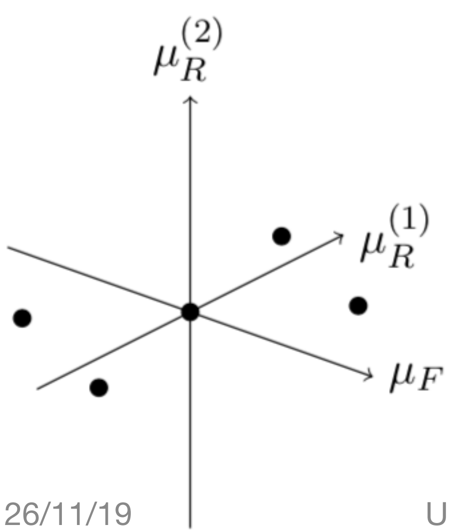
*i, j from **same process***



$$\text{cov}_{\text{th},ij} = \frac{1}{2}\left\{\Delta_i(+,+)\Delta_j(+,+) + \Delta_i(-,-)\Delta_j(-,-)\right\}$$

μ_F, μ_R fully correlated

*i, j from **different processes***



$$\text{cov}_{\text{th},ij} = \frac{1}{4}\left\{(\Delta_i(+,+) + \Delta_i(-,-))(\Delta_j(+,+) + \Delta_j(-,-))\right\}$$

μ_F, μ_R fully uncorrelated

$\Rightarrow$ missing μ_F correlation

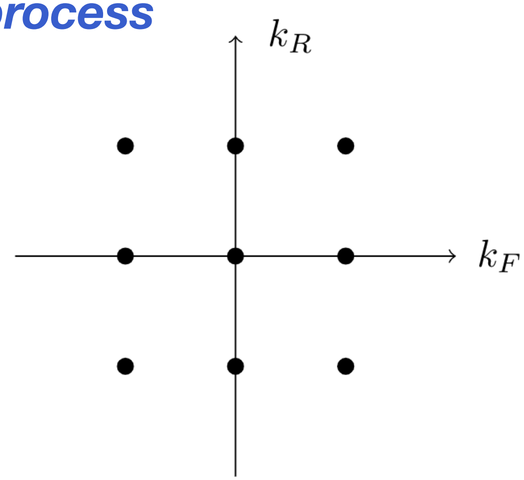
where

$$\Delta_i(+,+) = t_i(k_F = 2,\ k_R = 2) - t_i(k_F = 1,\ k_R = 1)$$

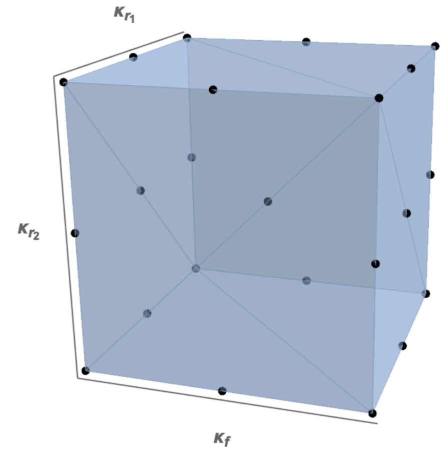
$$\Delta_i(-,-) = t_i\left(k_F = \frac{1}{2},\ k_R = \frac{1}{2}\right) - t_i(k_F = 1,\ k_R = 1)$$

More complex scale combinations: 9-pt

*i, j from **same process***



*i, j from **different processes***

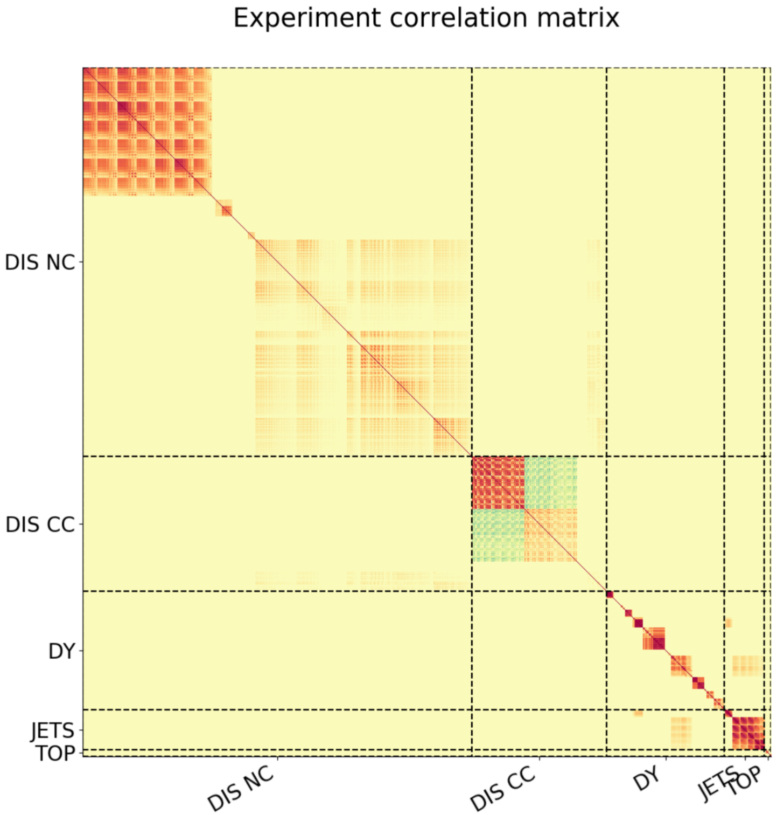


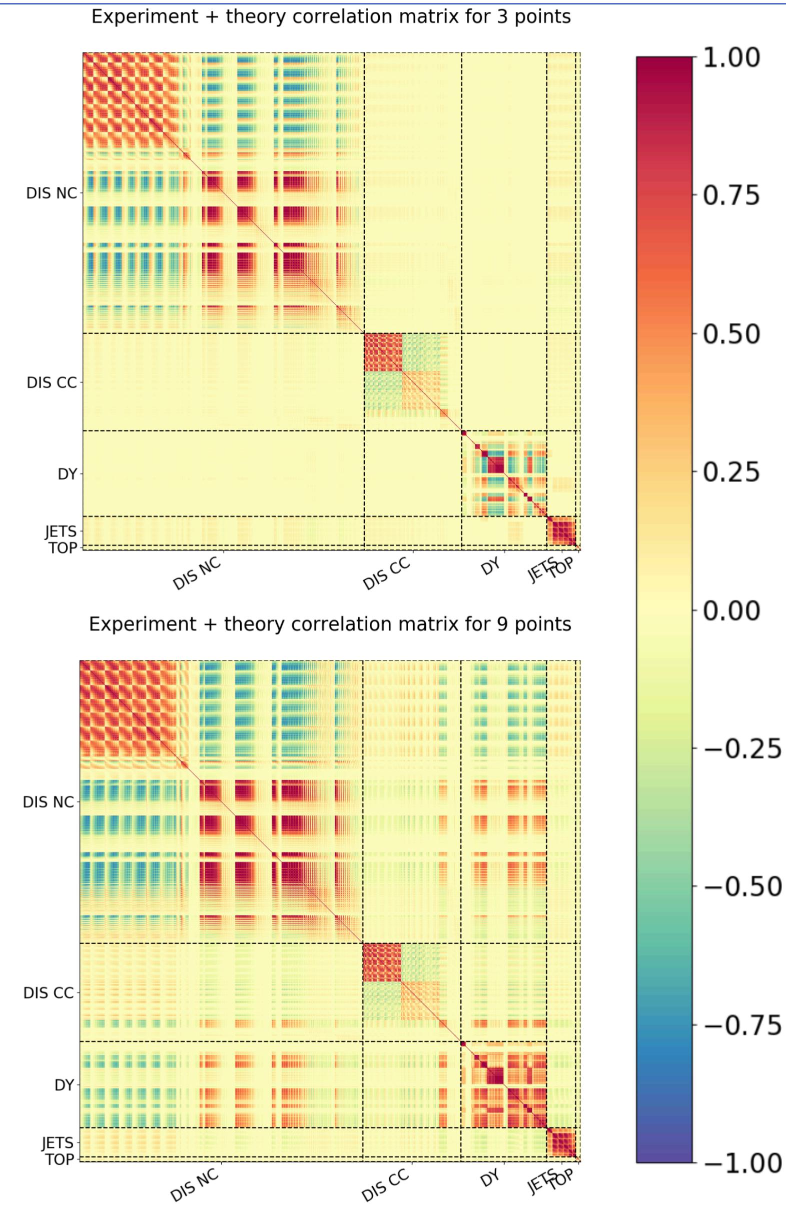
The more complex scale combination allows us to define **more complex correlation structure**:

- same process: μ_F, μ_R fully correlated
- different processes: μ_F fully correlated, μ_R fully uncorrelated

We expect this to produce a more **accurate** correlation structure, since we account for different α_s dependence in partonic cross sections and common DGLAP evolution

A theoretical covariance matrix for MHOUs





How can we **validate** and **compare** our theory covariance matrices?

Validation

- We validate **NLO cov$_{\text{th}}$** against exact result: **NNLO-NLO shift**

Findings:

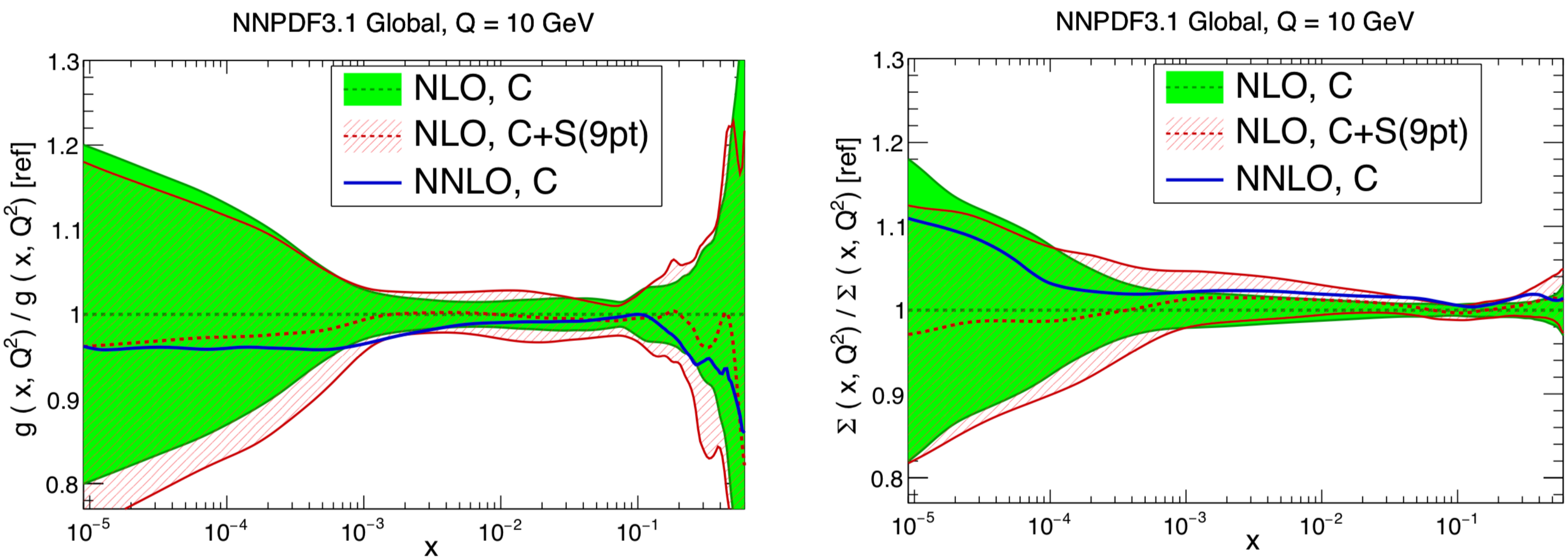
- For an *n*-pt prescription, the higher *n* is, the better the **cov$_{\text{th}}$** is able to describe the NNLO-NLO shift
- The extra points present in 9-pt vs 7-pt lead to 9-pt performing better

Use 9-pt in our PDF fits

NB: all results including MHOUs here are at NLO

Results: PDF fits with cov_{th}

- We use cov_{th} in both MC **sampling** (replica generation) and **fitting** (χ^2)

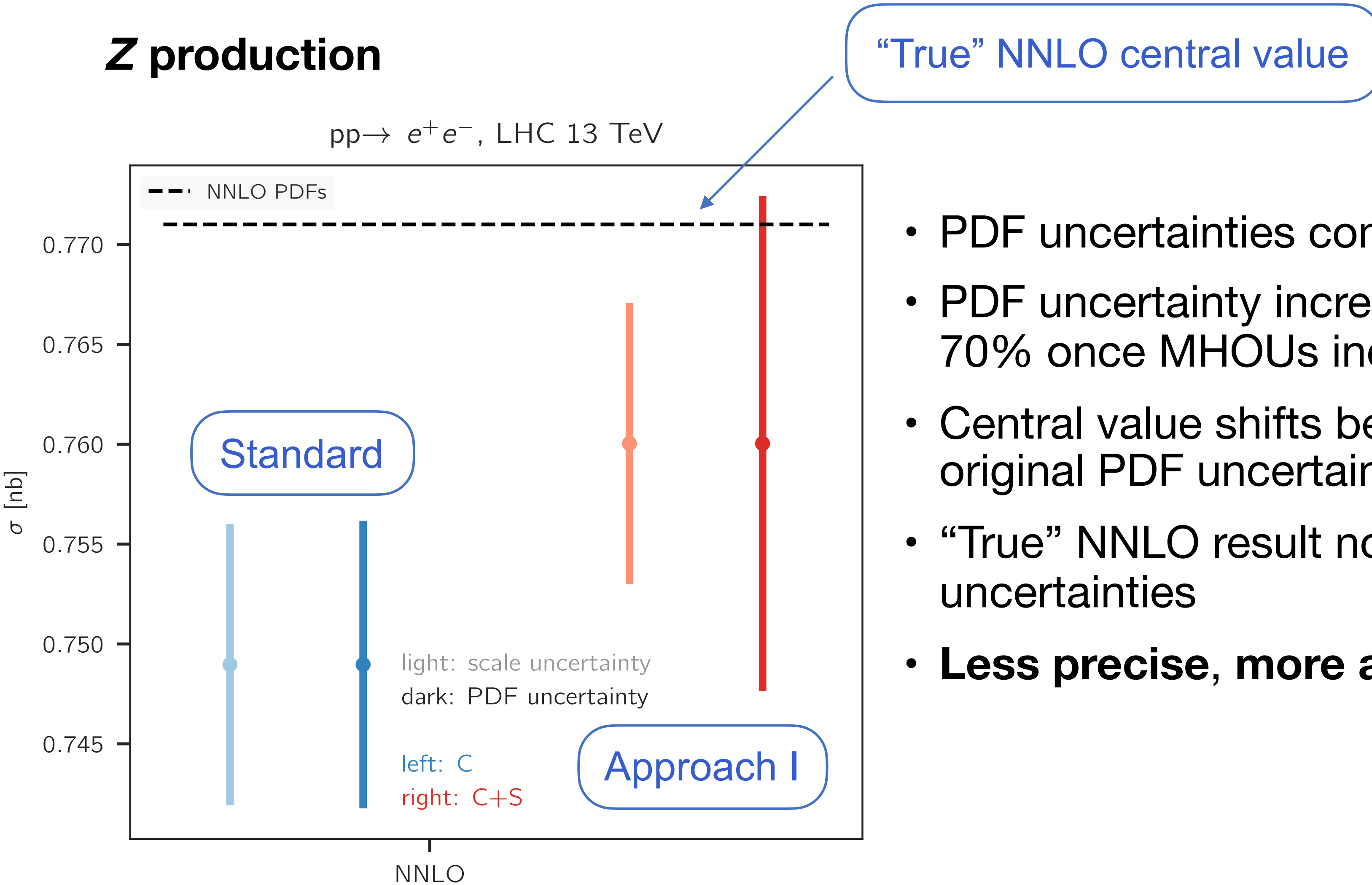


- Overall **small increase** in uncertainties (if at all): **tensions relieved**
- When NNLO-NLO shift large compared to PDF uncertainty, PDF shifts to account for this

$\Rightarrow$ **More reliable** PDF uncertainties

Results: Impact at the LHC

Z production



- PDF uncertainties compatible
- PDF uncertainty increases by 70% once MHOUs included
- Central value shifts beyond original PDF uncertainty
- "True" NNLO result now within uncertainties
- **Less precise, more accurate**

Approach I: Conclusions

- Systematically including MHOUs in PDFs is now important, and will become crucial
- A new framework for including MHOUs in PDFs has been developed, based on **fitting with a theory covariance matrix**
- This is **validated** against **NNLO-NLO shift**
- Using this we have produced the **first PDF fits including MHOUs**, which are **more consistent** with NNLO PDFs than standard NLO fits
- Framework is applicable to **all sources of theoretical uncertainty**

Approach II:
Monte Carlo scales uncertainties

In collaboration with M. Ubiali and Z. Kassabov

Monte Carlo scale uncertainties

Idea: sample from the space of scale variations for each PDF replica

Overcomes two limitations of the theory covariance matrix approach:

1. The user can **resample** the replicas
2. Can keep track of **correlation** between scales in observable prediction and scales in PDFs

Monte Carlo scale uncertainties

Idea: sample from the space of scale variations for each PDF replica

- Split data into N_p processes, assign one μ_F (fully correlated approx.) and N_p renormalisation scales to theory predictions for each replica
- Vary these scales. Again, $k_F, k_R \in \left(\frac{1}{2}, 1, 2\right)$
- Build set of N_{rep} replicas where **scale info. is recorded** (in LHAPDF files)

$\Rightarrow$ Experimental uncertainties and MHOUs propagated to PDFs

Monte Carlo scale uncertainties

- There are then 3^{N_p+1} scale combinations (729 for $N_p = 5$)
- Given $N_{\text{rep}} = 100$ for a normal PDF fit (~ 1 day per replica), impractical to fit same no. of replicas for each scale combination

$\Rightarrow$ Define **probability distribution** for sampling scale combinations

Monte Carlo scale uncertainties

- There are then 3^{N_p+1} scale combinations (729 for $N_p = 5$)
- Given $N_{\text{rep}} = 100$ for a normal PDF fit (~ 1 day per replica), impractical to fit same no. of replicas for each scale combination

$\Rightarrow$ Define **probability distribution** for sampling scale combinations

Define: $P(\mu = \xi) = \sum_{\text{all reps where } \mu=\xi} P(\omega)$, where $\omega \in (\mu_F, \mu_{R,1}, \ldots, \mu_{R,N_p})$

Define: $P(\mu_1 = \xi_1 \,|\, \mu_2 = \xi_2) = \dfrac{1}{P(\mu_2 = \xi_2)} \sum_{\text{all reps where } \mu_1=\xi_1,\ \mu_2=\xi_2} P(\omega)$

Symmetries

Choose symmetries, e.g.:

- For one process, probability of sampling replica invariant under switching factorisation and renormalisation scales, e.g. $P(\mu_F = x) = P(\mu_R = x)$
- Probability of sampling replica invariant under flipping variation for any scale (i.e. $\mu = 2 \leftrightarrow 0.5, \mu = 1 \leftrightarrow 1$)
- …

Free parameters

Under symmetries of the model, there are just **three free parameters**

$$a \equiv \frac{P(k_F = 1)}{P(k_F = 2)} = \frac{P(k_F = 1)}{P(k_F = \frac{1}{2})}$$

$$b \equiv \frac{P(k_R = 1 \,|\, k_F = 1)}{P(k_R = 2 \,|\, k_F = 1)} = \frac{P(k_R = 1 \,|\, k_F = 1)}{P(k_R = \frac{1}{2} \,|\, k_F = 1)}$$

$$c \equiv \frac{P(k_R = 2 \,|\, k_F = 2)}{P(k_R = \frac{1}{2} \,|\, k_F = 2)} = \frac{P(k_R = \frac{1}{2} \,|\, k_F = \frac{1}{2})}{P(k_R = 2 \,|\, k_F = \frac{1}{2})}$$

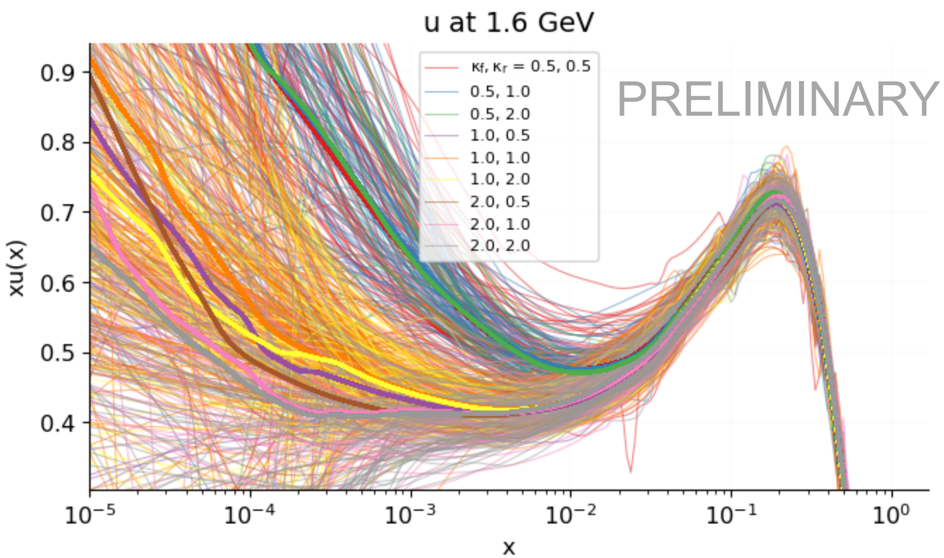
Interpretation:

- If μ_F and μ_R are totally **independent** then $a = b$, $c = 1$
- If μ_F and μ_R are fully **correlated** then $b, c \to \infty$

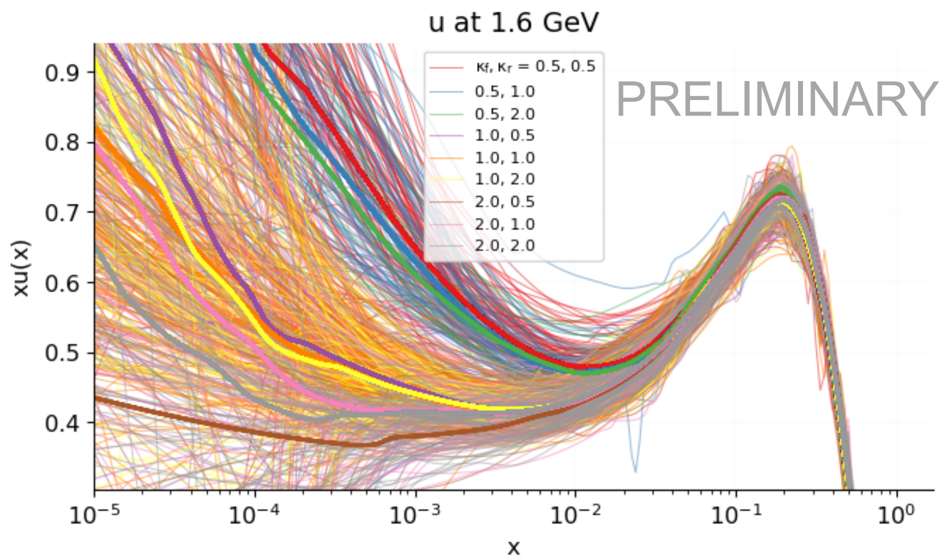
Preliminary results: PDFs

$$a = 2, b = \frac{10}{3}, c = 9$$

DIS CC

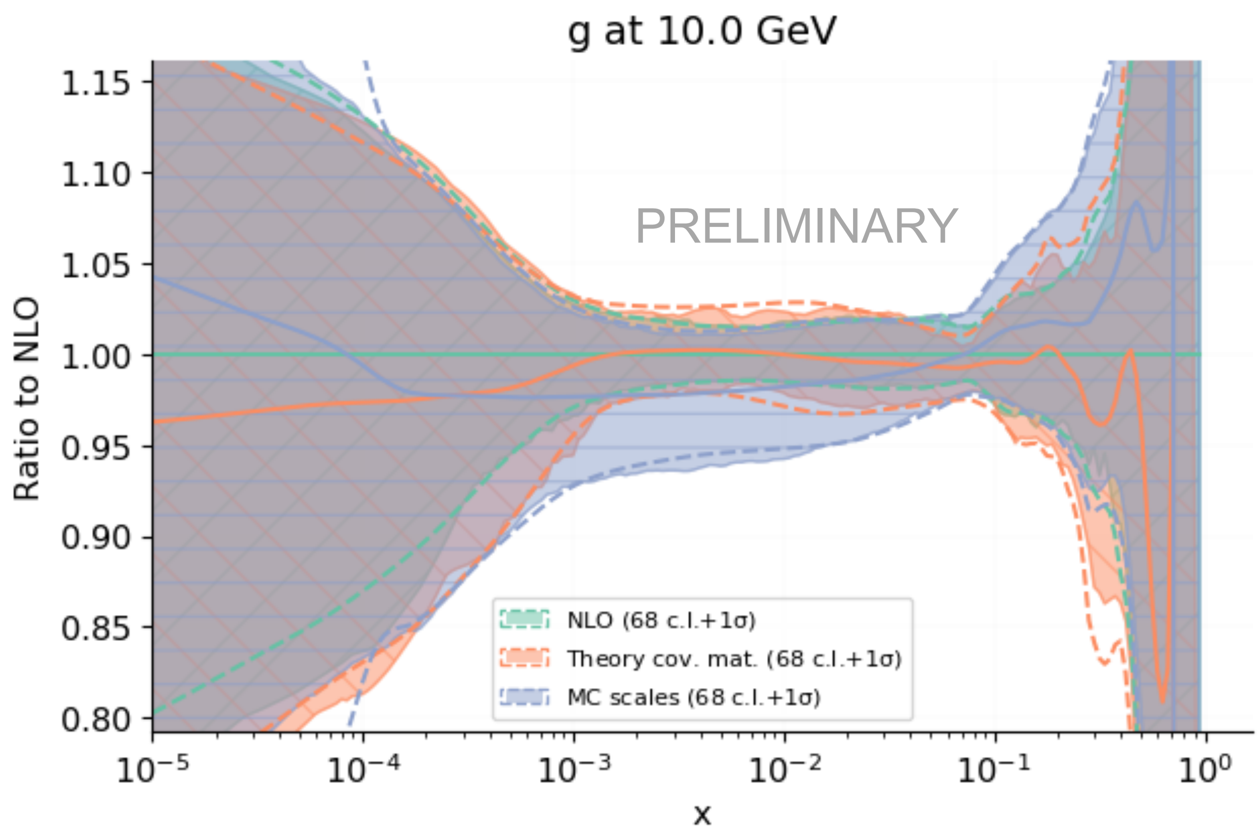


Jets



- We can plot PDF replicas and analyse the **scale dependence for each process**
- Can ask new questions: e.g. do certain scale choices for certain processes lead to bad fits?

Preliminary results: PDFs



$$a = 2$$
$$b = \frac{10}{3}$$
$$c = 9$$

- **Compatible PDFs** with theory cov. mat. and MC scales approaches
- MC scales leads to **larger uncertainties** in data regions → effect of MHOU not "integrated out" in each PDF replica

Computing cross sections

‘**Default**’ predictions:

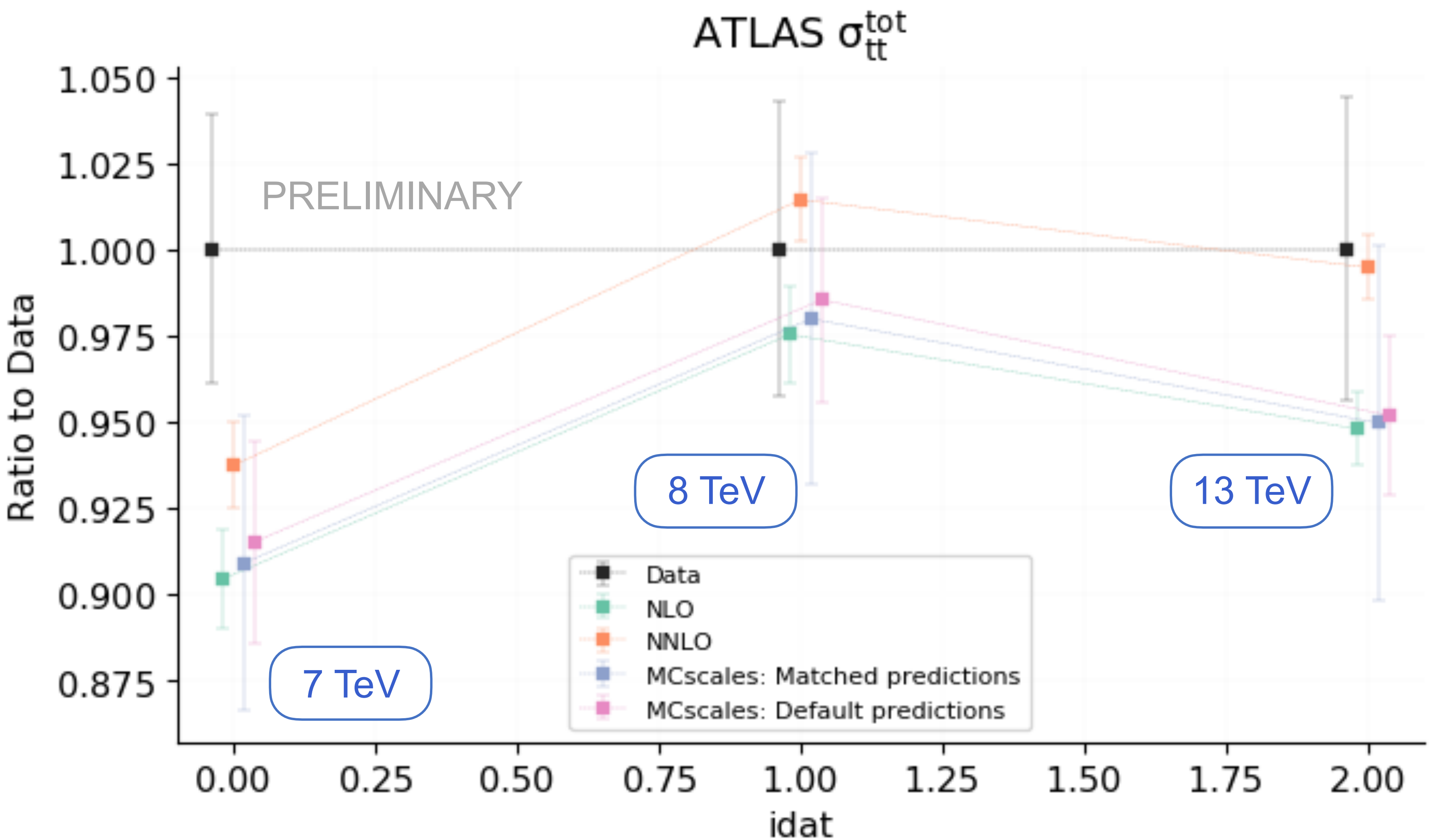
- Whatever scale choices in partonic cross section, convolute with all PDF replicas

‘**Matched**’ predictions:

- Combine pieces in **correlated** way
- Convolute PDF replicas with partonic cross section at **same scales**
- Generate combined scale variation + PDF (inc. MHOU) uncertainty

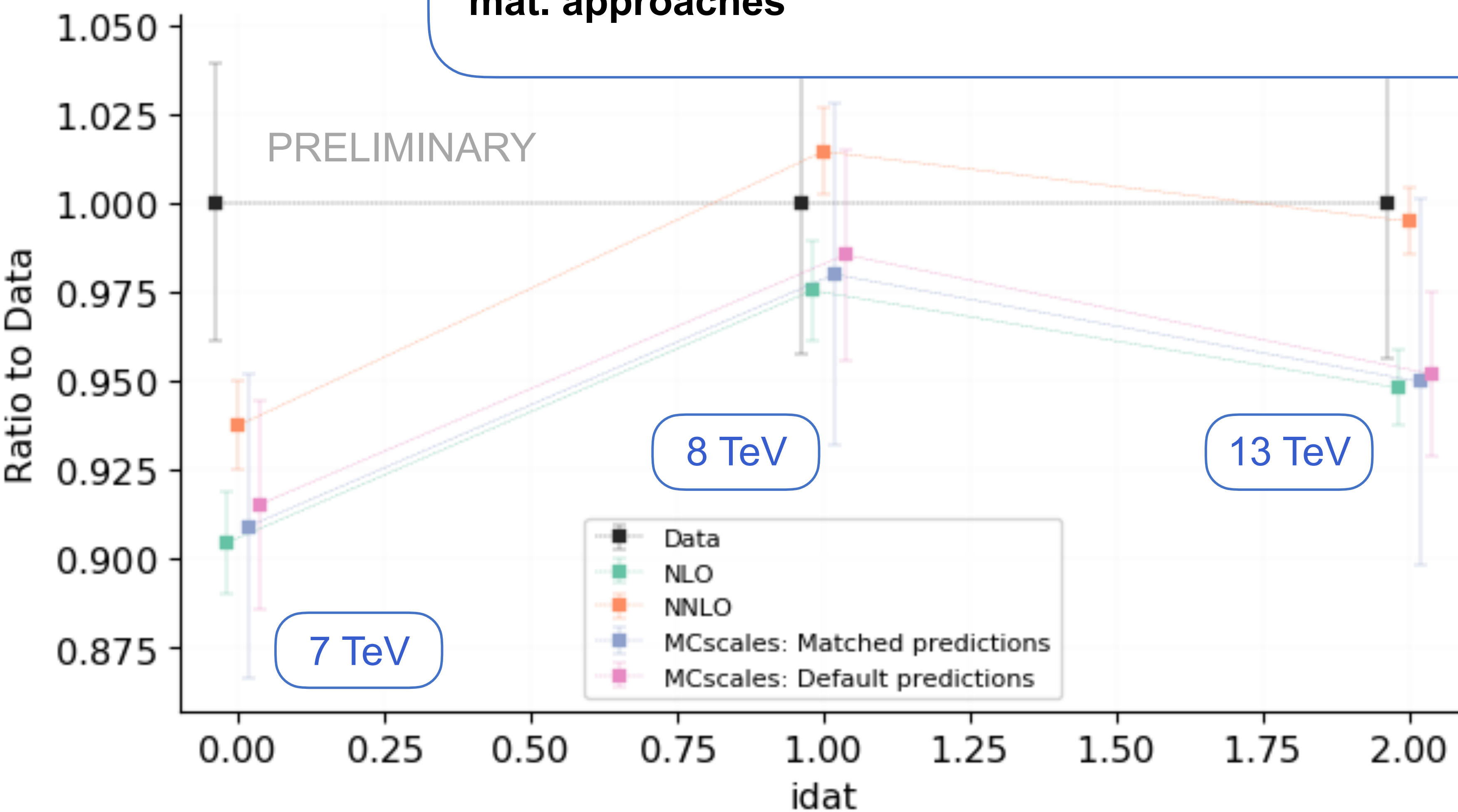
$$\sigma = \langle \sum_{\text{scale choices}} \hat{\sigma}(\mu_R = \mu_{R,\text{proc}}, \mu_F = \mu_{F,\text{proc}}) \otimes \mathscr{L}(\mu_F = \mu_{F,\text{proc}}, \mu_R = \mu_{R,\text{proc}}) \rangle$$

Preliminary results: cross sections



Preliminary re

- Increase in PDF uncertainties
- 'Matched predictions' gives reasonable estimate of MHOUs
- **More systematic study necessary to compare standard NLO with MC scales (default vs matched) and theory cov. mat. approaches**



Future work

- Develop MC scales approach by e.g. studying impact of choices of a, b, c
- Study differences between theory cov. mat. and MC scales.

  $\rightarrow$ Do they give similar results?

Refine each approach:

- Study impact of **process categorisation**
- **Decorrelate** μ_F by having independent variations for different PDFs (singlet vs non-singlet evolution)
- Produce **global NNLO fits with MHOUs** included - will be most state-of-the-art PDFs available

Thank you for listening!

Extra slides

The NNPDF approach

Guiding principles: introduce **minimal theoretical prejudice** into functional form of PDFs, and use **statistically sound error propagation**

1. Generate N_{rep} **'data replicas'** by Monte Carlo sampling according to distribution of exp. data and their uncertainties, correlations (defined by cov_{exp})
2. For each data replica, parametrise PDFs with **Neural Networks**
3. Fit N_{rep} **'PDF replicas'** using χ^2 as a figure of merit with certain **algorithm**

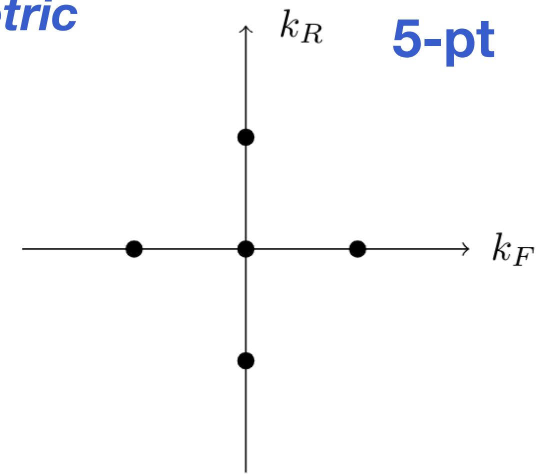
$$\chi^2 = (\text{data} - \text{theory})^T (\text{cov}_{\text{exp}})^{-1} (\text{data} - \text{theory})$$

$$\text{cov}_{\text{exp,ij}} = \rho_{ij}\,\sigma_i\,\sigma_j$$

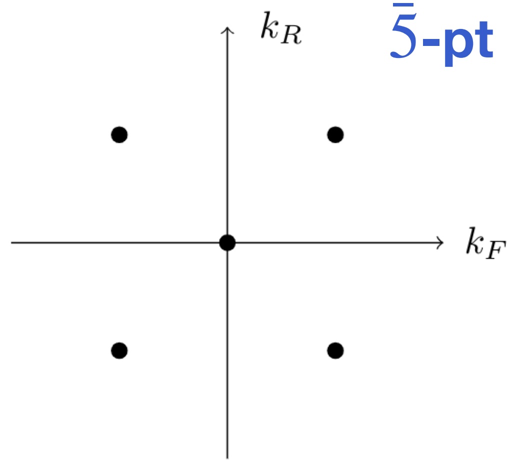
⇒ maximise agreement between data and theoretical predictions for each replica

Point prescriptions

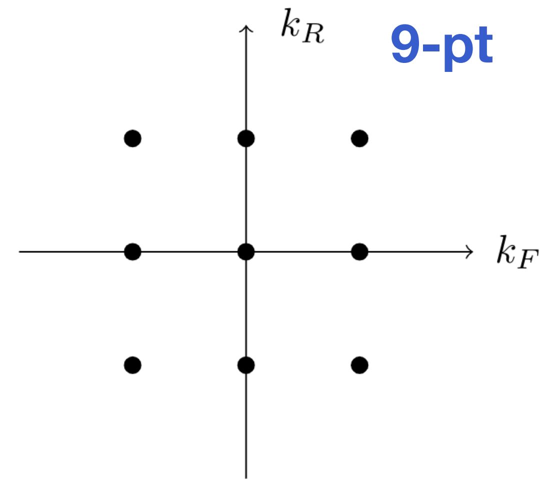
Symmetric



5-pt

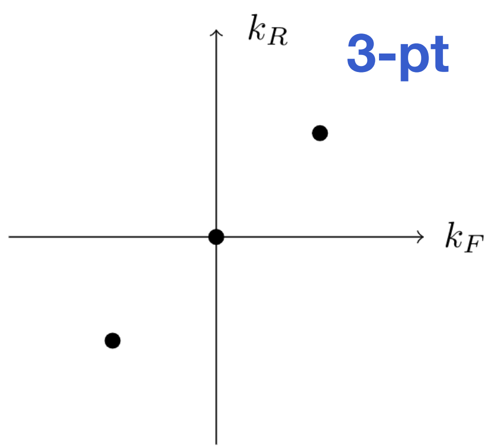


$\bar{5}$-pt

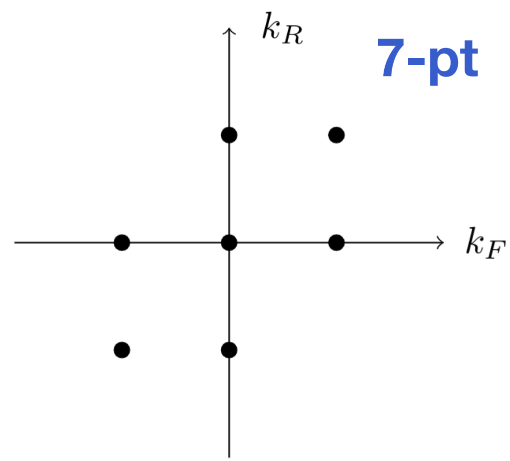


9-pt

Asymmetric



3-pt



7-pt

Data set and cuts

The following datasets are included in both `NNPDF31_nlo_as_0118_1000` and `190302_ern_nlo_central_163_global`:

- HERA I+II inclusive NC e^+p 920 GeV
- NMC p
- LHCb Z 940 pb
- CMS W rapidity 8 TeV
- D0 Z rapidity
- HERA I+II inclusive CC e^+p
- CDF Z rapidity
- ATLAS low-mass DY 2011
- CMS $\sigma_{tt}^{\rm tot}$
- HERA I+II inclusive NC e^+p 820 GeV
- CHORUS $\sigma_{CC}{}^{\bar{\nu}}$
- ATLAS W,Z 7 TeV 2011
- ATLAS HM DY 7 TeV
- ATLAS $\sigma_{tt}^{\rm tot}$
- BCDMS d
- BCDMS p
- LHCb $W,Z \to \mu$ 8 TeV
- CMS W asymmetry 840 pb
- HERA I+II inclusive NC e^+p 575
- NuTeV $\sigma_c{}^{\bar{\nu}}$
- HERA I+II inclusive NC e^+p 460
- D0 $W \to e\nu$ asymmetry
- HERA I+II inclusive CC e^-p
- D0 $W \to \mu\nu$ asymmetry
- NMC d/p
- HERA $\sigma_c^{\rm NC}$
- SLAC d
- CMS Drell-Yan 2D 7 TeV 2011
- LHCb $W,Z \to \mu$ 7 TeV
- LHCb $Z \to ee$ 2 fb
- ATLAS $t\bar{t}$ rapidity y_t
- NuTeV $\sigma_c{}^{\nu}$
- SLAC p
- ATLAS Z p_T 8 TeV ($p_T{}^{ll}, M_{ll}$)
- CHORUS $\sigma_{CC}{}^{\nu}$
- ATLAS Z p_T 8 TeV ($p_T{}^{ll}, y_{ll}$)
- CMS jets 7 TeV 2011
- CMS $t\bar{t}$ rapidity $y_{t\bar{t}}$
- HERA I+II inclusive NC e^-p
- CMS Z p_T 8 TeV ($p_T{}^{ll}, y_{ll}$)
- CMS W asymmetry 4.7 fb
- ATLAS W,Z 7 TeV 2010
- ATLAS jets 2011 7 TeV

Changes to cuts:

$$Q^2_{\rm min} = 3.49 \to 13.96 \text{ GeV}^2$$

Intersection of NLO, NNLO cuts

The following datasets are included in `NNPDF31_nlo_as_0118_1000` but not in `190302_ern_nlo_central_163_global`:

- ATLAS jets 2.76 TeV
- CMS $W + c$ ratio
- DY E886 $\sigma^p_{\rm DY}$
- ATLAS jets 2010 7 TeV
- CMS jets 2.76 TeV
- HERA H1 $F_2{}^b$
- DYE 866 $\sigma^d_{\rm DY}/\sigma^p_{\rm DY}$
- CMS $W + c$ total
- DY E605 $\sigma^p_{\rm DY}$
- CDF Run II k_t jets
- HERA ZEUS $F_2{}^b$

Data removed:

- Fixed target Drell-Yan
- Bottom structure function
- Jets without exact NNLO theory
- W+charm

Validation: uncertainties + correlations

- We validate cov_{th} against exact result: **NNLO-NLO shift**

- cov_{th} is **positive semi-definite** (eigenvalues > 0 or 0)
- Eigenvalue of covariance matrix is variance in direction of eigenvector
- Eigenvalue = 0 $\Rightarrow$ no variance/**shift** predicted by cov_{th} in direction of eigenvector
- Define **angle**, θ, of matrix as angle between shift and proportion of shift that is contained within **non-zero eigenvectors**

$$0^{\circ} \leq \theta \leq 90^{\circ}$$

$\theta = 0^{\circ}$: cov_{th} predicts variation in same directions as shift

Validation: uncertainties + correlations

3-pt

Per **data set**: $0.14^\circ \leq \theta \leq 73.5^\circ$

Per **process**:

Process	Angle, θ
DIS NC	54°
DIS CC	36°
DY	39°
Jets	24°
Top	12°

Global: $\theta = 52^\circ$

9-pt

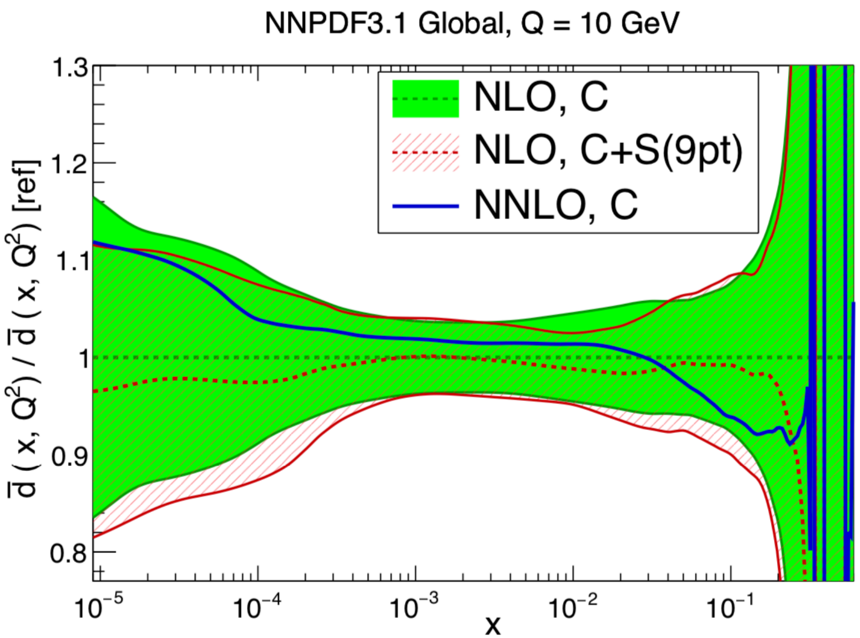
Per **data set**: $0.00^\circ \leq \theta \leq 24.6^\circ$

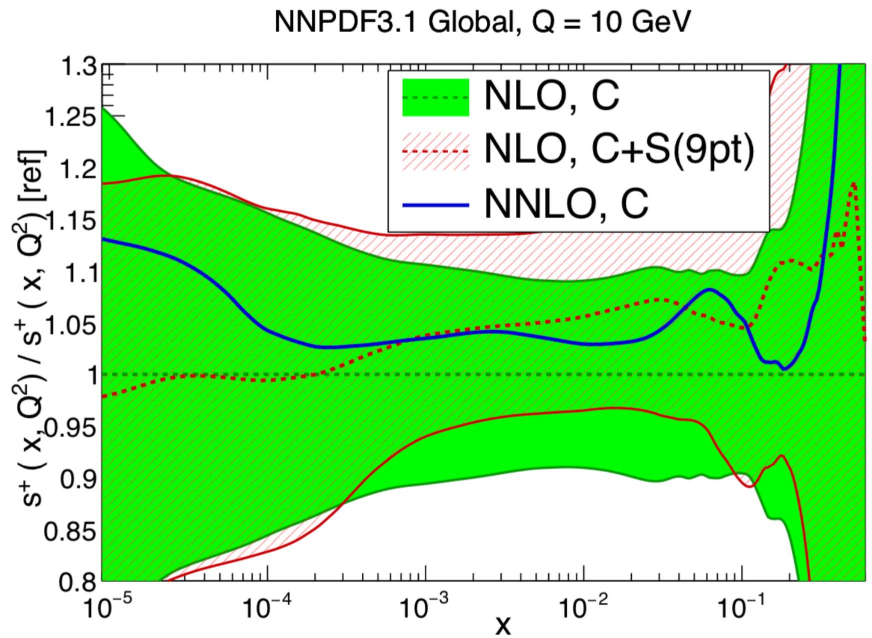
Per **process**:

Process	Angle, θ
DIS NC	32°
DIS CC	16°
DY	22°
Jets	14°
Top	3°

Global: $\theta = 26^\circ$

Results: PDF fits with cov_{th}



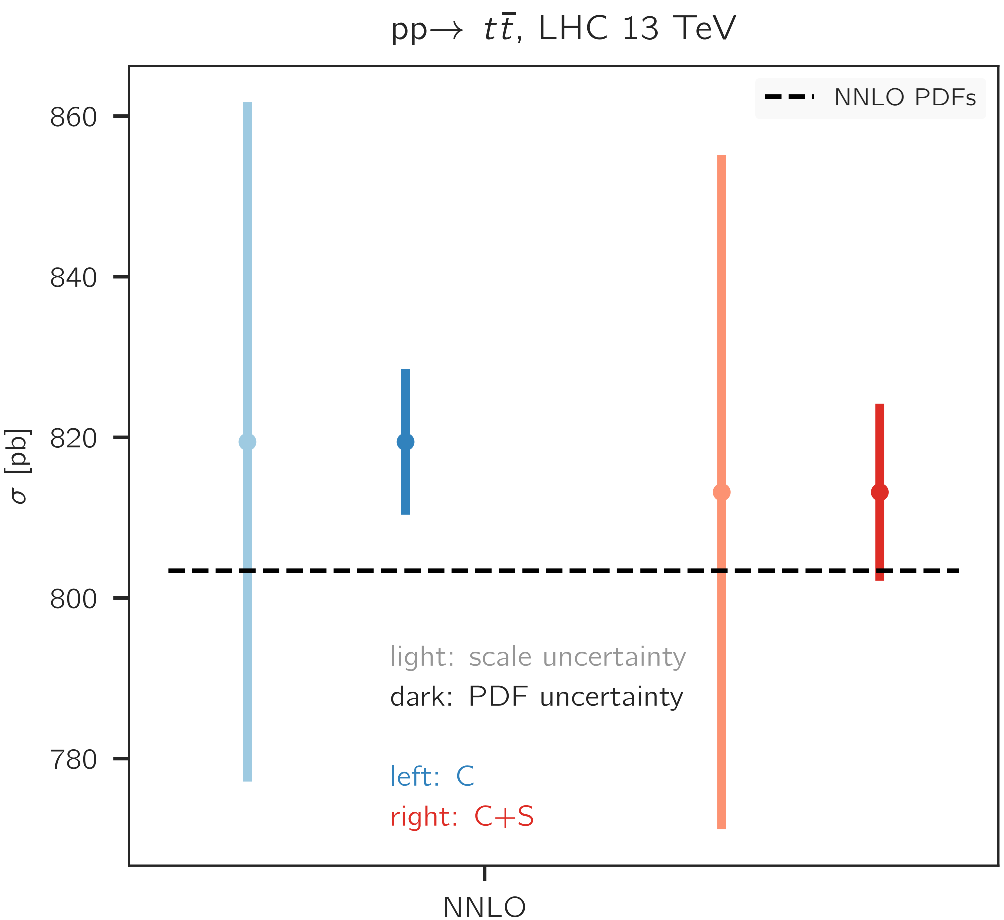


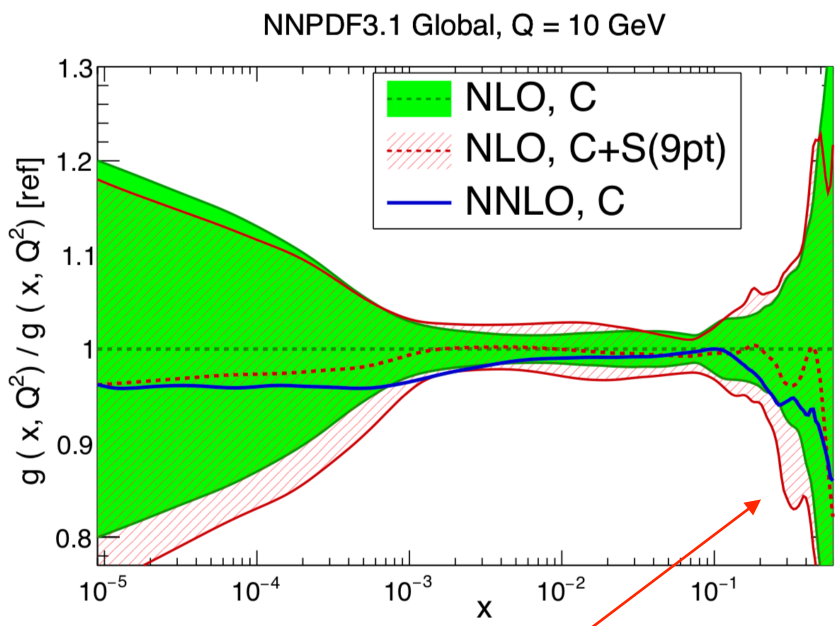
Results: Impact at the LHC

- Recommended method for combining partonic cross section with PDFs in theory cov. mat. approach: **proceed as normal**
    1. Use DGLAP evolution with central scale choice (μ_F variation accounted for elsewhere)
    2. Compute PDF uncertainty as normal, by convoluting all PDF replicas with partonic cross section at central scales: this now includes MHOUs
    3. Estimate MHOU on partonic cross section by using scale variations, can e.g. use a point prescription

Results: Impact at the LHC

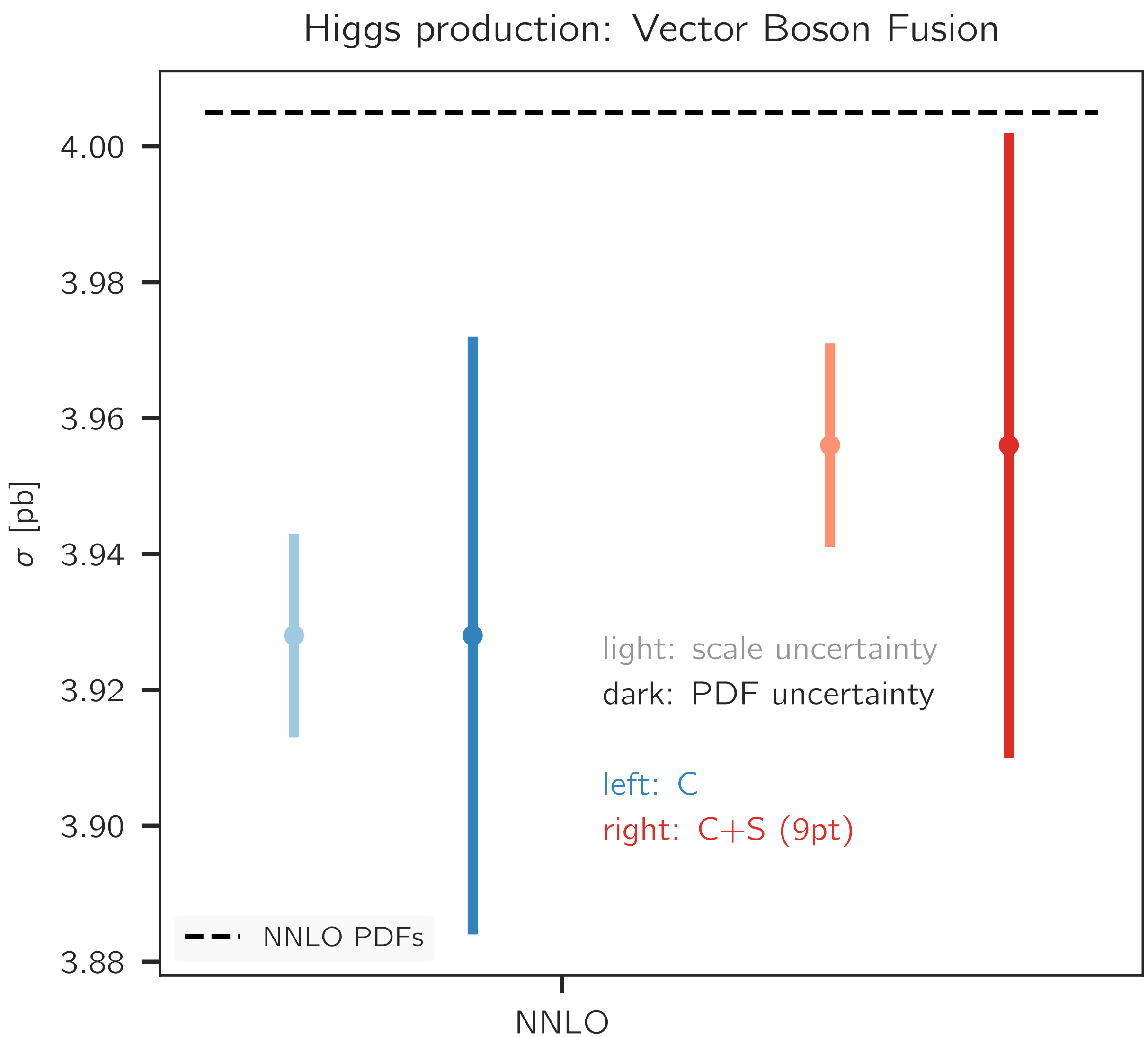
Top pair production





Relevant quantity:
Gluon PDF at $x \sim 0.3$

Results: Impact at the LHC



Sampling model - symmetries

1. For one process, probability invariant under exchange of μ_F and μ_R

$$P(\mu_F = \xi) = P(\mu_{R,i} = \xi) \qquad \forall\ i = 1, \ldots, N_p$$

2. Conditional probabilities symmetric

$$P(\mu_F = \xi_x \,|\, \mu_{R,i} = \xi_y) = P(\mu_{R,i} = \xi_x \,|\, \mu_F = \xi_y) \qquad \forall\ i = 1, \ldots, N_p$$

3. Probability symmetric under flipping of upper and lower variations

$$P(\mu_F = 2,\ \mu_{R,1} = 1,\ \mu_{R,2} = \frac{1}{2}, \ldots) = P(\mu_F = \frac{1}{2},\ \mu_{R,1} = 1,\ \mu_{R,2} = 2, \ldots)$$

Sampling model - symmetries

4. Renormalisation scales are not directly dependent on each other

$$P(\mu_{R,i} = \xi_i \,|\, \mu_F = \xi_F, \mu_{R,j} = \xi_j) = P(\mu_{R,i} = \xi_i \,|\, \mu_F = \xi_F)$$

$$\forall\ i, j = 1, \ldots, N_p$$

5. Symmetry between renormalisation scales

$$P(\mu_{R,i} = \xi) = P(\mu_{R,j} = \xi)$$

$$\forall\ i, j = 1, \ldots, N_p$$

$$P(\mu_{R,i} = \xi \,|\, \mu_F = \xi_\mu) = P(\mu_{R,j} = \xi \,|\, \mu_F = \xi_\mu)$$

Monte Carlo scale uncertainties

- μ_R variations independent so we write:

$$P(\mu_F = \xi_F, \ldots, \mu_{R,N_p} = \xi_{R,N_p}) = P(\mu_F = \xi_F) \prod_{i=1}^{N_p} P(\mu_{R,i} = \xi_{R,i} \mid \mu_F = \xi_F)$$

$P(\mu_F = \xi_F) \downarrow$ 3 $\qquad P(\mu_{R,i} = \xi_{R,i} \mid \mu_F = \xi_F) \downarrow$ 9

- Four normalisation constraints:

$$\sum_{\xi \in \frac{1}{2},1,2} P(\mu = \xi) = 1 \qquad \sum_{\xi \in \frac{1}{2},1,2} P(\mu = \xi \mid \mu_F = \xi_F) = 1 \qquad 12 \to 8$$

- Symmetry when flipping upper and lower variations: 4 more $\quad 8 \to 4$
- Symmetry when flipping μ_F and μ_R in probability: 1 more $\quad 4 \to 3$

Preliminary results: cross sections