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## PAST PRESENT AND FUTURE CHALLENGES IN THE DETERMINATION OF THE STRUCTURE OF THE PROTON

### LECTURE II

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# Exercise I: Z contribution

- Show that, in the Parton model, considering also the contribution of a virtual Z boson and its interference with the photon one obtains:

$$F_2^{\gamma, Z}(x) = x \sum_{i=1}^{n_f} c_i [q_i(x) + \bar{q}_i(x)]$$

$$F_3^{\gamma, Z}(x) = \sum_{i=1}^{n_f} d_i [q_i(x) - \bar{q}_i(x)]$$

Where

$$c_i = e_i^2 - 2e_i V_{eZ} V_{iZ} P_Z + (V_{eZ}^2 + A_{eZ}^2)(V_{iZ}^2 + A_{iZ}^2) P_Z^2$$

$$d_i = -2e_i A_{eZ} A_{iZ} P_Z + 4V_{eZ} A_{eZ} V_{iZ} A_{iZ} P_Z^2$$

$$P_Z = \frac{Q^2}{(Q^2 + M_Z^2)(4s_w^2 c_w^2)}$$



$$c_w = \cos \theta_w$$

$$s_w = \sin \theta_w$$

# Solution I

I In the lectures we considered

$$\frac{1}{2} \sum_{\mu\nu} \left| \begin{array}{c} \gamma^* \\ \gamma \end{array} \right|^2 = \frac{e^2}{2} \text{Tr} [\hat{p} \gamma^\mu \hat{p}' \gamma^\nu]$$

Add to hadronic tensor

$$\frac{1}{2} \sum_{\mu\nu} \left| \begin{array}{c} \gamma^* \\ \gamma \end{array} \right|^2 = \frac{e^2 k_z^2}{2} \text{Tr} [\hat{p} \gamma^\mu (\bar{N}_{iz} + \gamma^s \bar{Q}_{iz}) \hat{p}' \gamma^\nu (\bar{N}_{iz} + \gamma^s \bar{Q}_{iz})]$$

$$= \frac{e^2}{8 S_W^2 C_W^2} \left[ (\bar{N}_{iz}^2 + \bar{Q}_{iz}^2) \text{Tr} [\hat{p} \gamma^\mu \hat{p}' \gamma^\nu] + 2 \bar{N}_{iz} \bar{Q}_{iz} \text{Tr} [\hat{p} \gamma^\mu \hat{p}' \gamma^\nu \gamma^s] \right]$$

$\downarrow$   $4[\hat{p}^\mu \hat{p}'^\nu + \hat{p}^\nu \hat{p}'^\mu - \hat{p}^\mu \hat{p}'^\mu]$        $\downarrow$   $4i \sum_{\alpha\beta\mu\eta} \hat{p}_\alpha \hat{p}'_\beta$   
 to when contracted with anti-sym tensor

Similar contribution in leptonic tensor

$$\bar{N}_{iz} \rightarrow \bar{N}_{ez} \quad \hat{p}_z^2$$

$$\bar{Q}_{iz} \rightarrow \bar{Q}_{ez}$$

and propagator  $\frac{1}{(Q^2 + M_z^2)^2}$

$$\left\{ \begin{array}{c} z^* \\ z \end{array} \right\} \quad \left\{ \begin{array}{c} z^* \\ z \end{array} \right\}$$

have different signs in front of anti-sym. part

# Solution I

$$\begin{aligned} & \frac{1}{2} \sum_i \left( \overbrace{\gamma^*}^{\gamma^*} \rightarrow \right)^* \left( \overbrace{\gamma^*}^{\gamma^*} \rightarrow \right) \\ &= -\frac{1}{2} e^2 k_F k_B T_R [ \hat{P} \gamma^\mu (U_{iz} + Q_{iz} \gamma^5) \hat{P}' \gamma^\mu ] \\ &\approx -\frac{e^2 c_J}{4 S_W C_W} [ U_{iz} T_R [ \hat{P} \gamma^\mu \hat{P}' \gamma^\mu ] + Q_{iz} T_R [ \hat{P} \gamma^\mu \hat{P}' \gamma^\mu \gamma^5 ] ] \end{aligned}$$

Similar contribution in  
leptonic tensor

$$U_{iz} \rightarrow N_{ez}$$

$$Q_{iz} \rightarrow \alpha_{ez}$$

and propagator

$$\frac{Q^4}{(Q^2 + M_Z^2)} \frac{1}{Q^4}$$

$\underbrace{\qquad\qquad\qquad}_{P_Z}$

## Exercise II: Paschos-Wolfenstein relation

- Show that, in the Parton model, considering a (anti)neutrino-initiated DIS process on a deuteron target – assuming SU(2) isospin symmetry  $u_n(x)=d_p(x)$  and  $d_n(x) = u_p(x)$  – the ratio R

$$R = \frac{\sigma_{NC}(\nu) - \sigma_{NC}(\bar{\nu})}{\sigma_{CC}(\nu) - \sigma_{CC}(\bar{\nu})}$$

NC (mediated by Z) and CC (mediated by  $W^{+/-}$ ), assuming strange and anti-strange to be equal in the target, is independent of Parton Distribution Functions and can be used to determine the Weinberg angle  $\theta_W$

$$R = \frac{1}{2} \left( \frac{1}{2} - \sin \theta_w^2 \right)$$

You may use (without deriving it) the result (and set  $c, c\bar{c} = 0$ )

$$F_2^{W^-} = 2x(u + \bar{d} + \bar{s} + c),$$

$$F_3^{W^-} = 2x(u - \bar{d} - \bar{s} + c),$$

$$F_2^{W^+} = 2x(d + \bar{u} + \bar{c} + s),$$

$$F_3^{W^+} = 2x(d - \bar{u} - \bar{c} + s),$$

# Exercise II: Paschos-Wolfenstein relation



TESTS FOR NEUTRAL CURRENTS IN NEUTRINO REACTIONS



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## ABSTRACT

Neutral currents predicted by weak interaction models of the type discussed by Weinberg may be detected in neutrino reactions. Limits on the ratio  $R$  of  $\sigma(\nu + N \rightarrow \nu + x)$  to  $\sigma(\nu + N \rightarrow \mu^- + x)$  are obtained independent of any dynamical assumption. For the total cross-section for high energy neutrinos, we find  $R \geq 0.18$ , provided the Weinberg mixing angle satisfies  $\sin^2 \theta_w \leq 0.33$ . For the production of a single  $\pi^0$  we find  $R' \geq 0.50$  contrasted with the experimental result  $R' \leq 0.14$  using only the assumption of (3, 3) resonance dominance. Applications are also given to anti-neutrino reactions.

# Solution II

$$\text{II) } \begin{array}{c} \bar{u}_e \quad u_e \\ \swarrow \quad \searrow \\ z^* \end{array} = \bar{U}'(u) \frac{-i\gamma_5}{4C_W} \gamma^\mu (1 - \gamma_5) U(u)$$

$$L_{\mu\nu}(u) = \frac{1}{8C_W^2} \left( \text{Tr} [k \gamma^\mu k \gamma^\nu] - \text{Tr} [\bar{k} \gamma^\mu k \gamma^\nu \gamma_5] \right)$$

$$\begin{array}{c} \bar{u}_e \quad u_e \\ \swarrow \quad \searrow \\ z^* \end{array} = \bar{U}'(u) \frac{-i\gamma_5}{4C_W} \gamma^\mu (1 - \gamma_5) U(u)$$

$$L_{\mu\nu}(\bar{U}) = \frac{1}{8C_W^2} \left( \text{Tr} [k' \gamma^\mu k' \gamma^\nu] - \text{Tr} [\bar{k}' \gamma^\mu k' \gamma^\nu \gamma_5] \right)$$

Because

$$d\sigma = \frac{1}{2s} \frac{q_{\mu\nu}^2 q_{\lambda\tau}^2}{(Q^2 + M_V^2)^2} L_{\mu\nu} W^{\mu\nu}(u\pi) \frac{d^3 k'}{(2\pi)^3 2E}$$

$\hookrightarrow M_V^4 \quad \text{for } Q^2 \ll M_V^2$

$$d\sigma(U) - d\sigma(\bar{U}) \propto \underbrace{[U - L_{\mu\nu}(\bar{U})]}_{\downarrow} \rightarrow \frac{q_{\mu\nu}^2 q_{\lambda\tau}^2}{M_V^4} W^{\mu\nu}(u\pi) \frac{d^3 k'}{(2\pi)^3 2E}$$

Only non-zero contribution when contracted with antisymmetric  $W_{\mu\nu}$

# Solution II

$$[L_{\mu\nu}(U) - L_{\mu\nu}(\bar{U})] W^{\mu\nu} = - \frac{1}{2(\tilde{p}\cdot q)c_w^2} k_\alpha w_\beta \epsilon^{\alpha\beta\mu\nu} \epsilon_{\rho\sigma\mu\nu} \tilde{p}^\rho q^\sigma F_3(x)$$

$$= \frac{ME}{c_w^2} \times (2-y) \times F_3^N(x)$$

$$\Rightarrow \frac{d^2\sigma}{dx dy} = \frac{G_F^2 c_w^2 M_n E_n}{2\pi^2} \times y (2-y) F_3^{NC}(x)$$

$$\int dy dx .. = \frac{G_F^2 c_w^2}{3\pi^2} E_n M_n \int dx \times F_3(x)$$

with  $F_3(x) \propto \frac{1}{2} \left( \frac{1}{2} - \frac{4}{3} S_w^2 \right) (U - \bar{U} + C - \bar{C}) + \frac{1}{2} \left( \frac{2}{3} S_w^2 - \frac{1}{2} \right) (D - \bar{D} + S - \bar{S})$

using  $U^P = D^N$      $D^P = U^N$      $\Rightarrow$      $U^P + D^P = U^N + D^N \Rightarrow U = d$   
 $\bar{U} = \bar{d}$

$$S = \bar{S}$$

$$C = \bar{C} \approx 0$$

$$\rightarrow F_3^{NC}(x) \propto \left( \frac{1}{2} - S_w^2 \right) [U(x) - \bar{U}(x)]$$

# Solution II

Charged Current

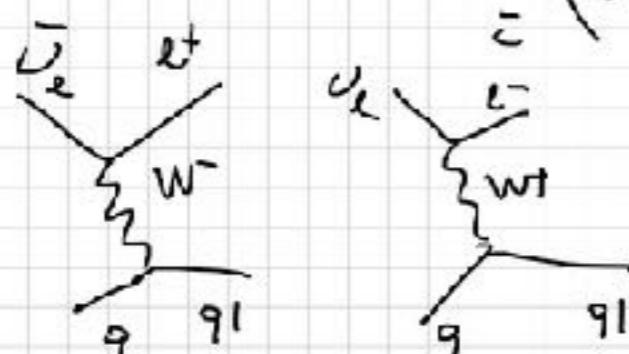
$$\hat{p}^* \quad \hat{p}^+ \quad \bar{U}_{kP} \left[ -\frac{iq}{2E_2} Y^N(n-s) U_{kj} \right] U_{jP}$$



CKM Matrix (Consider only  $e$  flavor)

$$U_{kj} = \begin{pmatrix} d & s \\ \bar{d} & \bar{s} \\ \bar{s} & c \\ -s \sin \theta_c & c \cos \theta_c \end{pmatrix} = \begin{pmatrix} 1 + \theta_c^2 & \theta_c \\ -\theta_c & 1 + \theta_c^2 \end{pmatrix}$$

expand in  $\theta_c$  and keep only  $O(1)$



$$\Rightarrow F_3^{W^+} = 2x(d - \bar{v} + s - \bar{c})$$

$$F_3^{W^-} = 2x(u - \bar{d} - \bar{s} + c)$$

Analogously to NC

$$\sigma_{cc}(u) - \sigma_{cc}(\bar{u}) = \frac{2G_F^2}{3\pi^2} E_\nu M_N \int x F_3^{cc}(x) dx$$

$$F_3^{cc}(x) = \frac{1}{2} (u + d + s + c - \bar{u} - \bar{d} - \bar{s} - \bar{c}) \rightarrow (u - \bar{u})$$

some hypothesis

$$= \frac{2G_F^2}{3\pi^2} E_\nu M_N \int dx (u(x) - \bar{u}(x))$$

## Solution II

taking ratio

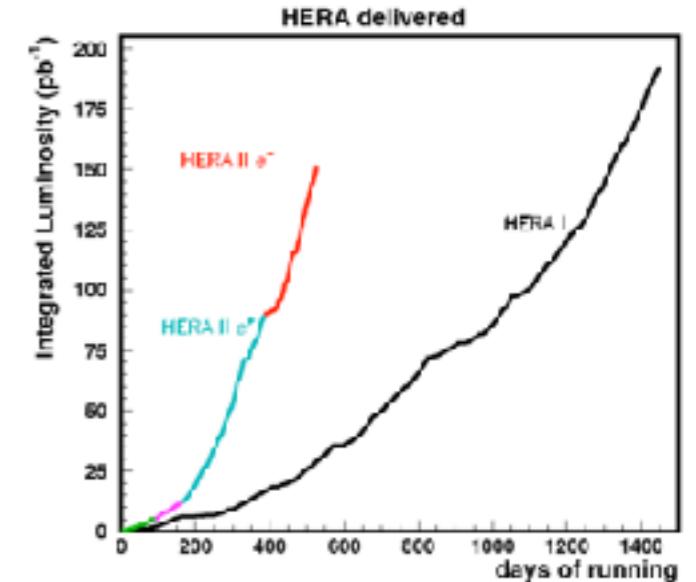
$$R = \frac{\frac{1}{2} (1 - 2 \sin^2 \omega) \int_0^1 dx [u(x) - \bar{u}(x)] dx}{2 \int_0^1 dx [u(x) - \bar{u}(x)]}$$

$$= \frac{1}{2} \left( \frac{1}{2} - \sin^2 \omega \right)$$

# Outline

- First lecture (Monday)
  - Motivation: the big picture
  - Parton Model and QCD
  - **Collinear Factorisation**
- **Second lecture (Tuesday)**
  - Experimental Data
  - Disentangling proton's components
  - Heavy quarks and photons
- Third lecture (Wednesday)
  - Fits and methodology
  - The NNPDF approach
- Fourth lecture (Thursday)
  - New frontiers

# The HERA collider



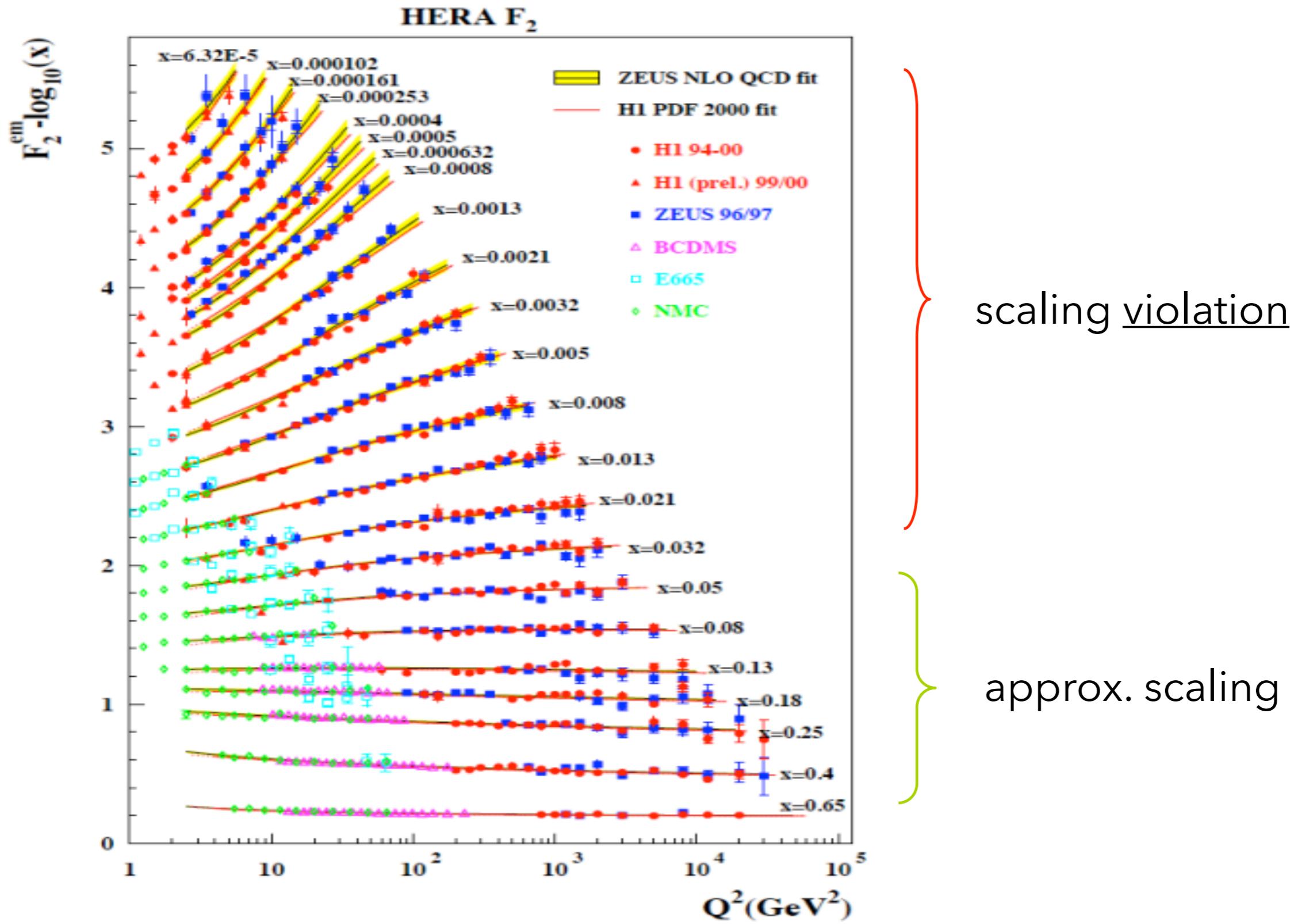
1992-2007

$$\sqrt{S} = 318 \text{ GeV}$$

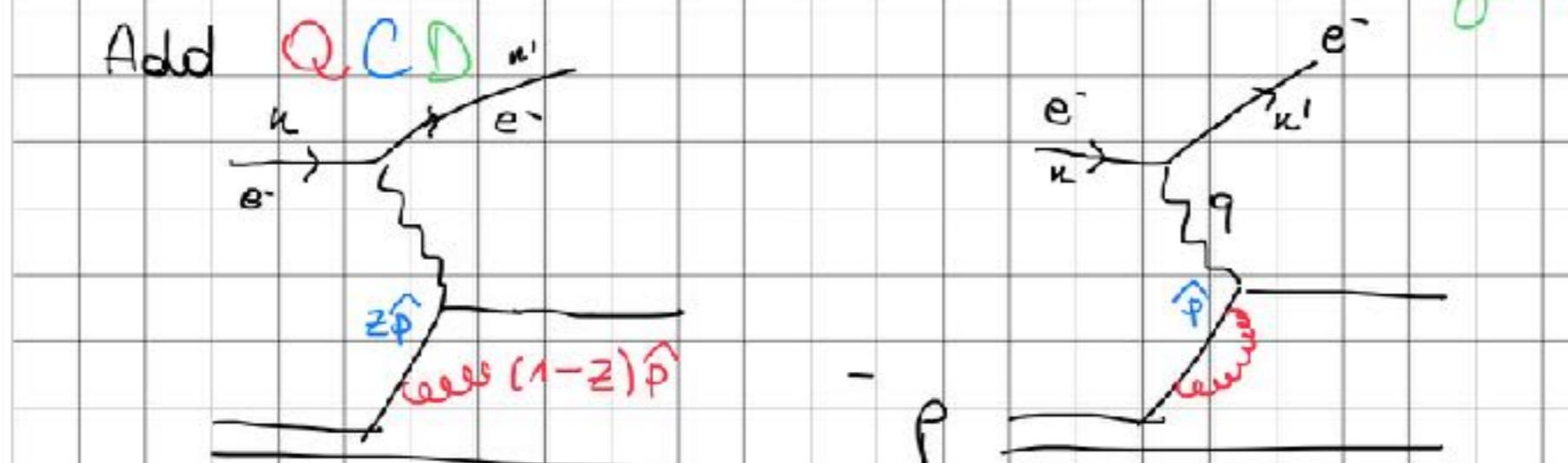
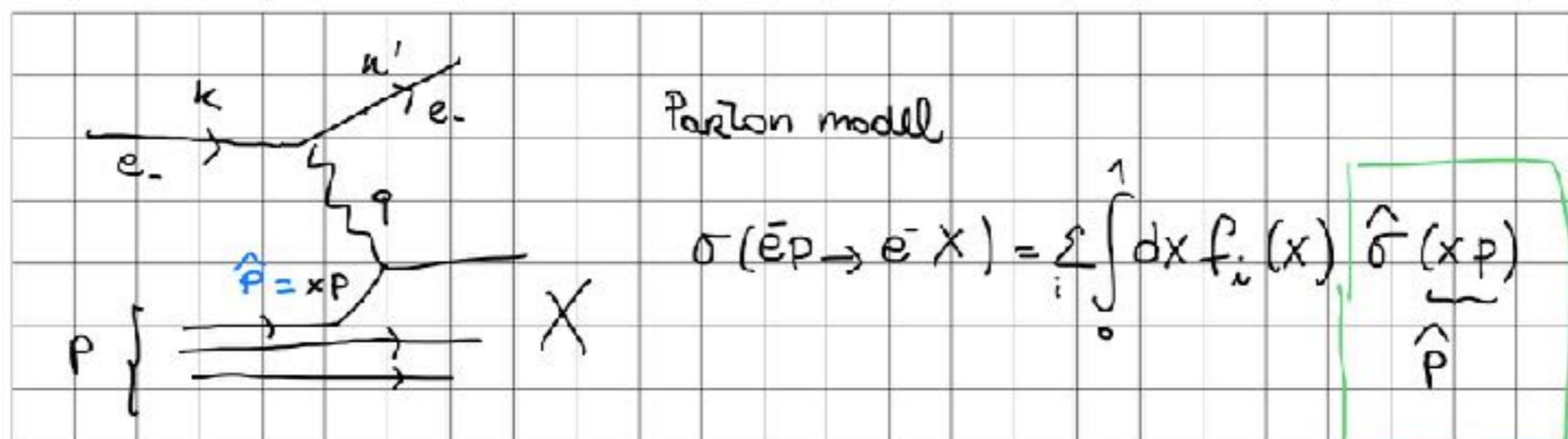
$$E_e = 27.5 \text{ GeV}$$

$$E_p = 920 \text{ GeV}$$

# Scaling violation



# QCD and improved parton model



Real emission

$$\sigma_q^{(n)}(p) = \frac{d\sigma}{2\pi} C_F \int_0^1 \frac{dz}{1-z} \int_{|u_T^\ell|}^{|u_T^\ell|_{max}} \frac{d|u_T^\ell|}{|u_T^\ell|} \frac{1+\xi^\ell}{|u_T^\ell|} \hat{\sigma}_q^0(zp)$$

# QCD and improved parton model

divergences!

Soft

$\mathbb{R} \rightarrow \mathbb{N}$

Repetitive  $\varepsilon \rightarrow 0$

## COLLINEAR

$$|k\gamma_i| \rightarrow 0$$

Regularize  $\lambda \rightarrow 0$

$$\hat{\sigma}_q^{(n)}(\rho) = \frac{\alpha_s}{2\pi} C_F \int_0^1 \frac{dz}{1-z} \int \frac{dk_F}{|k_F^\perp|} (1+z^2) \hat{\sigma}_q^{(n)}(zp)$$

## Adding visual corrections

$$-\frac{\hat{\sigma}_q^{(0)}}{2\pi} \left( p \right) \frac{d_S}{C_F} \int_0^{\infty} \frac{dz}{z-z} \int_{z_c}^{\frac{|k_T^\ell|_{\max}}{1-\varepsilon}} \frac{dk_T^\ell}{|k_T^\ell|} \left( 1+\varepsilon \right)$$

the soft singularity cancels  
 $\Rightarrow \epsilon \rightarrow 0$

$$\hat{\sigma}_q^{(1)}(p) = \frac{ds}{2\pi} C_F \int_0^1 \frac{dz}{1-z} \int_{z^E}^{1/\kappa_T^2 \text{max}} \frac{d(\kappa_T^2)}{|\kappa_T^2|} (1+z^2) \left[ \hat{\sigma}_q^{(0)}(zp) - \hat{\sigma}_q^{(0)}(p) \right]$$

Left with COLLINEAR divergence

# QCD and improved parton model

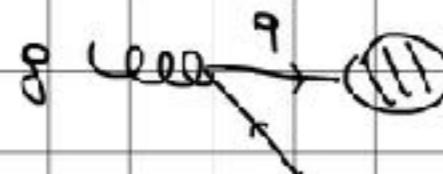
Still left with COLLINEAR divergence!

Introduce  $\mu_F$  to split integration

$$\int_{\lambda^2}^{k_T^2 \text{ max}} \frac{dk_T^2}{|k_T^2|} \rightarrow \underbrace{\int_{\lambda^2}^{\mu_F^2} \frac{dk_T^2}{|k_T^2|}}_{\text{singular}} + \underbrace{\int_{\mu_F^2}^{k_T^2 \text{ max}} \frac{dk_T^2}{|k_T^2|}}_{\text{finite}}$$
$$\Rightarrow \hat{\sigma}_q(\hat{p}) = \hat{\sigma}_q^{(0)}(\hat{p}) + \hat{\sigma}_q^{(1)}(\hat{p}) \quad \text{universal function } P(q \rightarrow q)$$
$$= \hat{\sigma}_q^{(0)}(\hat{p}) + \frac{ds}{2\pi} \int_1^\infty dz P_{qq}(z) \hat{\sigma}_q^{(0)}(z\hat{p}) \log \frac{\mu_F^2}{\lambda^2} + \hat{\sigma}_{q,\text{reg}}^{(1)}(\hat{p}, \mu_F^2)$$

# QCD and improved parton model

Of course quark can come from gluon



Doing the whole calculation, we get

$$\begin{aligned}\hat{\sigma}_g(p) &= \hat{\sigma}_g^{(1)}(p) \\ &= \frac{\alpha_s}{2\pi} \int_0^1 dz P_{qg}(z) \hat{\sigma}_q^{(0)}(zp) \log \frac{\mu_F^2}{z^2} + \hat{\sigma}_{g\text{reg}}^{(1)}(p, \mu_F^2)\end{aligned}$$

\* In the parton model formula

$$\sigma(p) = \int_0^1 dy [f_q(y) \hat{\sigma}_q(y p) + f_g(y) \hat{\sigma}_g(y p)]$$

# QCD and improved parton model

$$\begin{aligned}
 \sigma(p) = & \int_0^1 dy [f_q(y) \hat{\sigma}_q^{(0)}(y_p)] \\
 & + \frac{\alpha_s}{2\pi} \int_0^1 dy f_q(y) \int_0^1 dz \hat{\sigma}_q^{(0)}(yz_p) P_{qq}(z) \log \frac{\mu_F^2}{z^2} \\
 & + \frac{\alpha_s}{2\pi} \int_0^1 dy f_F(y) \int_0^1 dz \hat{\sigma}_F^{(0)}(yz_p) P_{qF}(z) \log \frac{\mu_F^2}{z^2} \\
 & + \int_0^1 dy f_q(y) \hat{\sigma}_{q, \text{reg}}^{(1)}(y_p, \mu_F^2) + \int_0^1 dy f_F(y) \hat{\sigma}_{F, \text{reg}}^{(1)}(y_p, \mu_F^2)
 \end{aligned}$$

↓ terms  $\propto \log \frac{\mu_F^2}{z^2}$  can be reabsorbed into redefinition of  $f_q$   
 $x = yz$

$$f_q(x, \mu_F^2) = \int_x^1 \frac{dy}{y} \left\{ f_q(y) \left[ \delta\left(1 - \frac{x}{y}\right) + \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{y}\right) \log \frac{\mu_F^2}{y^2} \right] \right. \\
 \left. + f_F(y) \left[ \frac{\alpha_s}{2\pi} P_{qF}\left(\frac{x}{y}\right) \log \frac{\mu_F^2}{y^2} \right] \right\}$$

# QCD and improved parton model

So that

$$\sigma(p) = \int_0^1 dx f_q(x, \mu_F^\epsilon) \hat{\sigma}_q(xp, \mu_F^\epsilon) + f_p(x) \hat{\sigma}_p(xp, \mu_F^\epsilon)$$



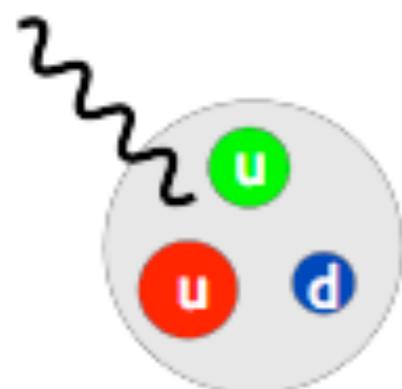
both depend on arbitrary  
FACTORIZATION scale

Note that however the dependence of  $f_{q,p}(x, \mu_F^\epsilon)$   
is totally fixed by perturbation theory

$$\begin{aligned} \mu^2 \frac{\partial f_q(x, \mu^2)}{\partial \mu^2} &= \frac{ds}{2\pi} \int_x^1 \frac{dy}{y} \left[ P_{qq} \left( \frac{x}{y} \right) f_q(y, \mu^2) \right. \\ &\quad \left. + P_{qF} \left( \frac{x}{y} \right) f_p(y, \mu^2) \right] \end{aligned}$$

# QCD and improved parton model

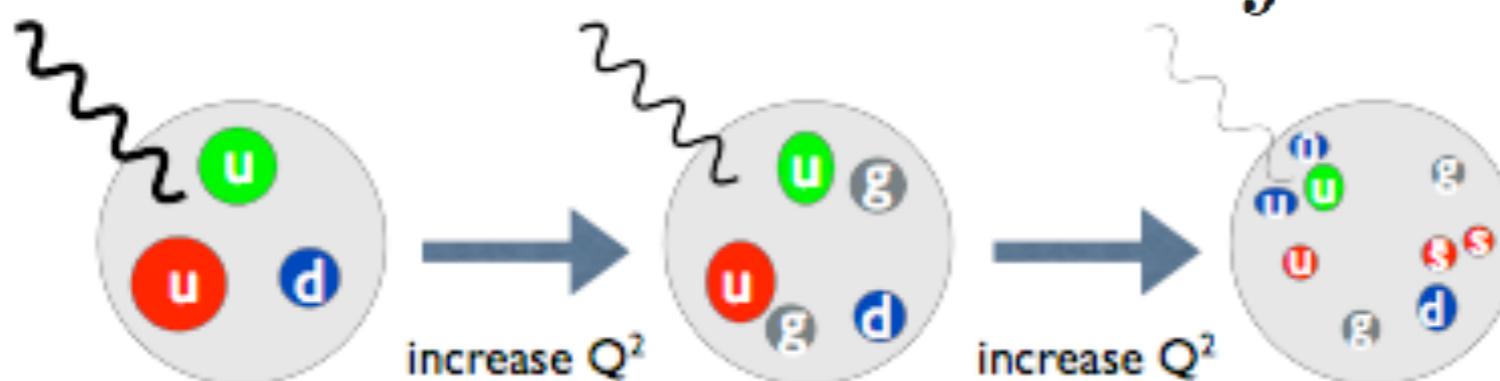
Parton model picture



$$\sigma = \int dx f_i^{(p)}(x) \sigma^{(0)}(xp)$$

QCD-improved parton model

$$\sigma = \int dx f_i^{(p)}(x, \mu_F^2) \sigma(xp, \mu_F^2)$$

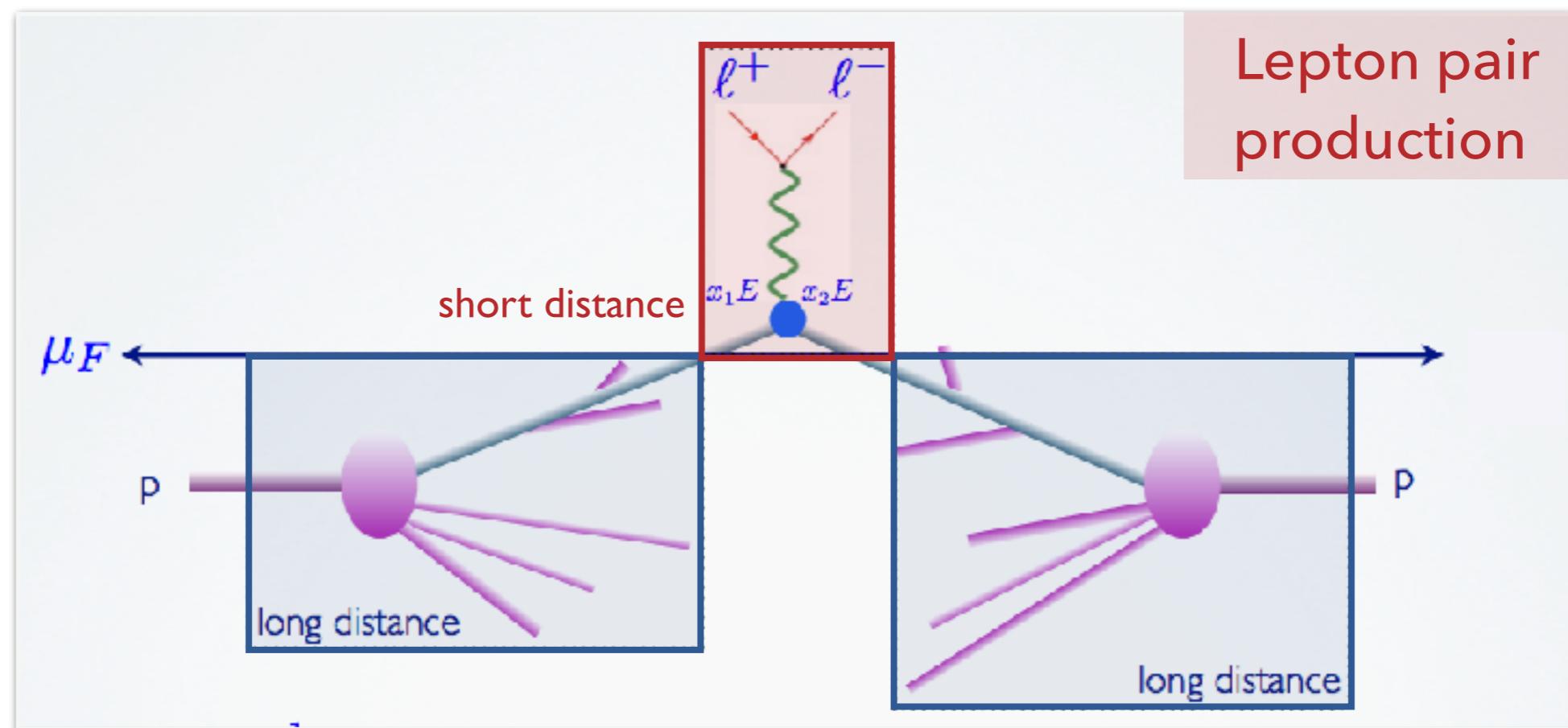


# Collinear factorisation theorem

# Collinear Factorisation Theorem

$$\frac{d\sigma_H^{ep \rightarrow ab}}{dX} = \sum_{i=-n_f}^{+n_f} \int_{x_B}^1 \frac{dz}{z} f_i(z, \mu_F) \frac{d\hat{\sigma}_i^{ei}}{dX}(zS, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$

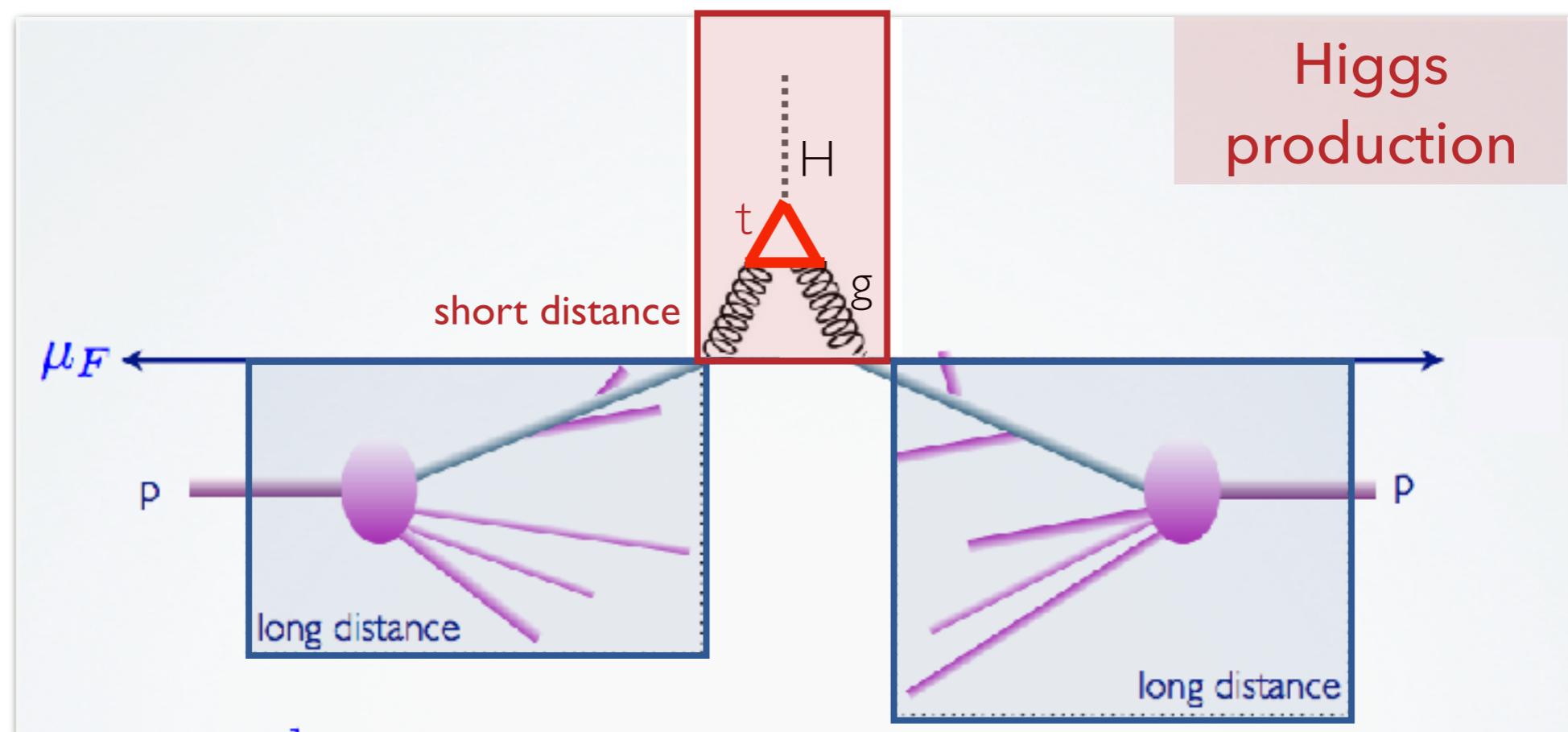
$$\frac{d\sigma_H^{pp \rightarrow ab}}{dX} = \sum_{i,j=-n_f}^{+n_f} \int_{\tau_0}^1 \frac{dz_1}{z_1} \frac{dz_2}{z_2} f_i(z_1, \mu_F) f_j(z_2, \mu_F) \frac{d\hat{\sigma}_i^{ij}}{dX}(zS, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$



# Collinear Factorisation Theorem

$$\frac{d\sigma_H^{ep \rightarrow ab}}{dX} = \sum_{i=-n_f}^{+n_f} \int_{x_B}^1 \frac{dz}{z} f_i(z, \mu_F) \frac{d\hat{\sigma}_i^{ei}}{dX}(zS, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$

$$\frac{d\sigma_H^{pp \rightarrow ab}}{dX} = \sum_{i,j=-n_f}^{+n_f} \int_{\tau_0}^1 \frac{dz_1}{z_1} \frac{dz_2}{z_2} f_i(z_1, \mu_F) f_j(z_2, \mu_F) \frac{d\hat{\sigma}_i^{ij}}{dX}(zS, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$

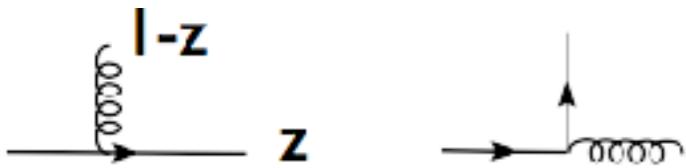


# DGLAP evolution equations

In analogy with running coupling, imposing that cross section does not depend on arbitrary scale  $\mu$ , get renormalisation group equations for PDFs

$$t = \log \frac{Q^2}{\mu_F^2} \quad \frac{d}{dt} \begin{pmatrix} q_i(x, t) \\ g(x, t) \end{pmatrix} = \frac{\alpha_s(t)}{2\pi} \int_x^1 \sum_{j=q,\bar{q}} \frac{d\xi}{\xi} \begin{pmatrix} P_{ij} \left( \frac{x}{\xi}, \alpha_s(t) \right) & P_{ig} \left( \frac{x}{\xi}, \alpha_s(t) \right) \\ P_{gj} \left( \frac{x}{\xi}, \alpha_s(t) \right) & P_{gg} \left( \frac{x}{\xi}, \alpha_s(t) \right) \end{pmatrix} \otimes \begin{pmatrix} q_j(\xi, t) \\ g(\xi, t) \end{pmatrix}$$

Dokshitzer, Gribov, Lipatov, Altarelli, Parisi equations

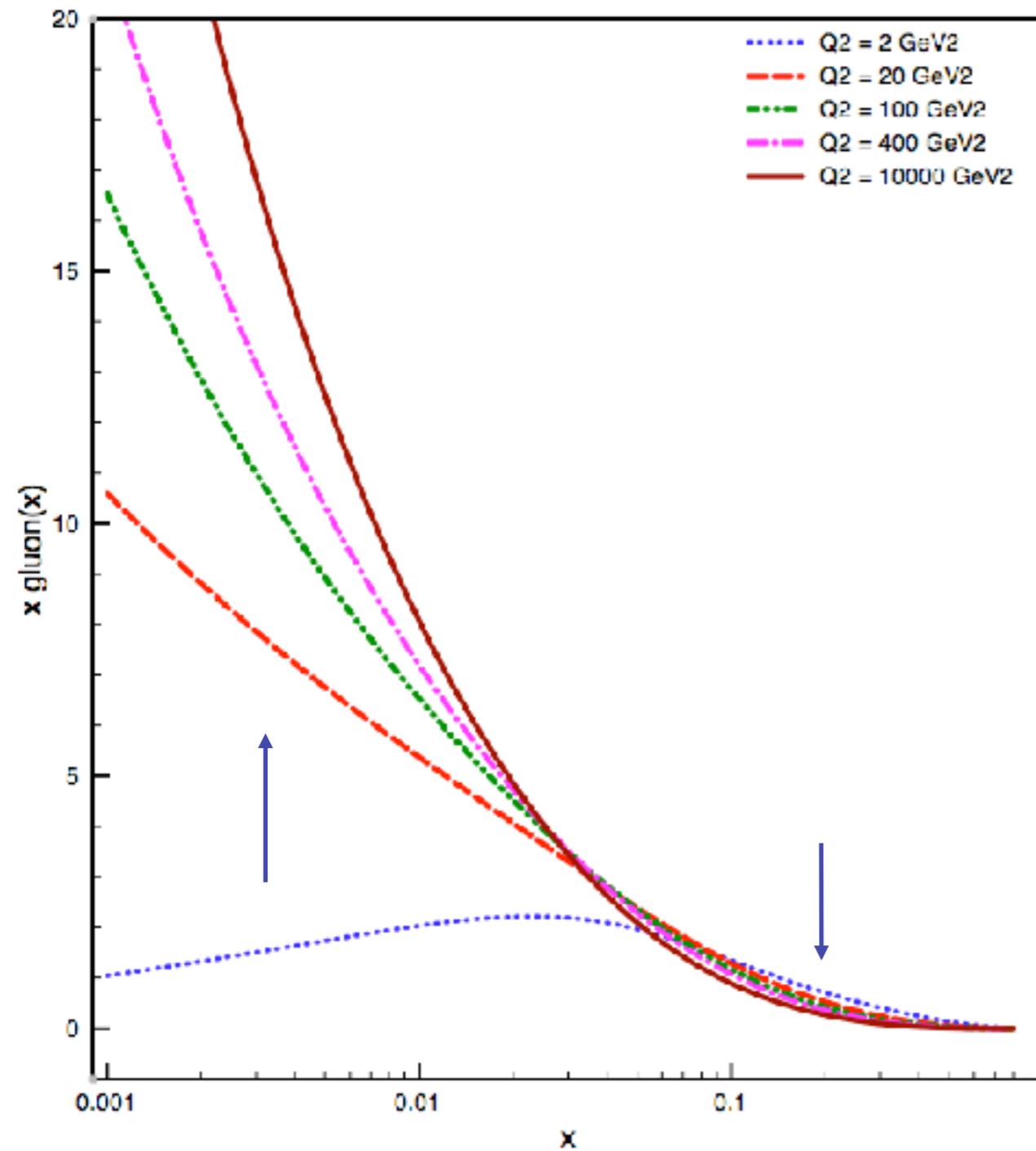


- Functional dependence on  $\mu^2$  is totally predicted by solving DGLAP evolution eqns



- Splitting functions known up to NNLO:
  - LO** Dokshitzer; Gribov, Lipatov; Altarelli, Parisi (1977)
  - NLO** Floratos, Ross, Sachrajda; Floratos, Lacaze, Kounnas, Gonzalez-Arroyo, Lopez, Yndurain; Curci, Furmanski Petronzio, (1981)
  - NNLO** - Moch, Vermaseren, Vogt, 2004

# DGLAP evolution equations



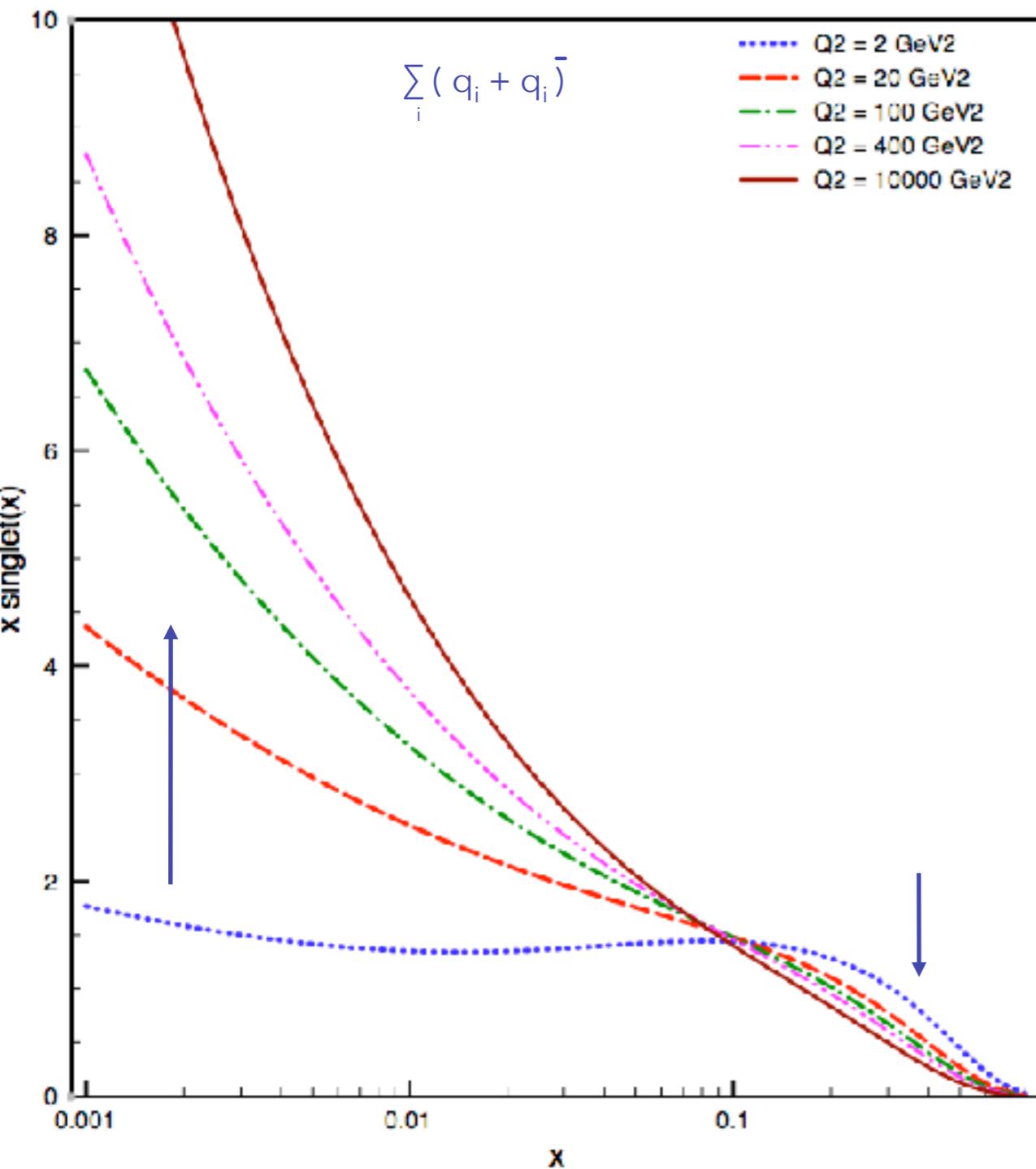
Gluon evolution

$$g(x, \mu^2) = \Gamma_{gq} \otimes \Sigma(x, \mu_0^2) + \Gamma_{gg} \otimes g(x, \mu_0^2)$$

$$\begin{aligned} P_{gq}^{(0)}(x) &= C_F \left[ \frac{1 + (1-x)^2}{x} \right] \\ P_{gg}^{(0)}(x) &= 2N \left[ \frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) \right] \\ &\quad + \delta(1-x) \frac{(11N - 4n_f T_R)}{6} \end{aligned}$$

- Both  $P_{gg}$  and  $P_{gq}$  diverge for  $x \rightarrow 0$
- Gluon is depleted at large  $x$

# DGLAP evolution equations



Singlet evolution

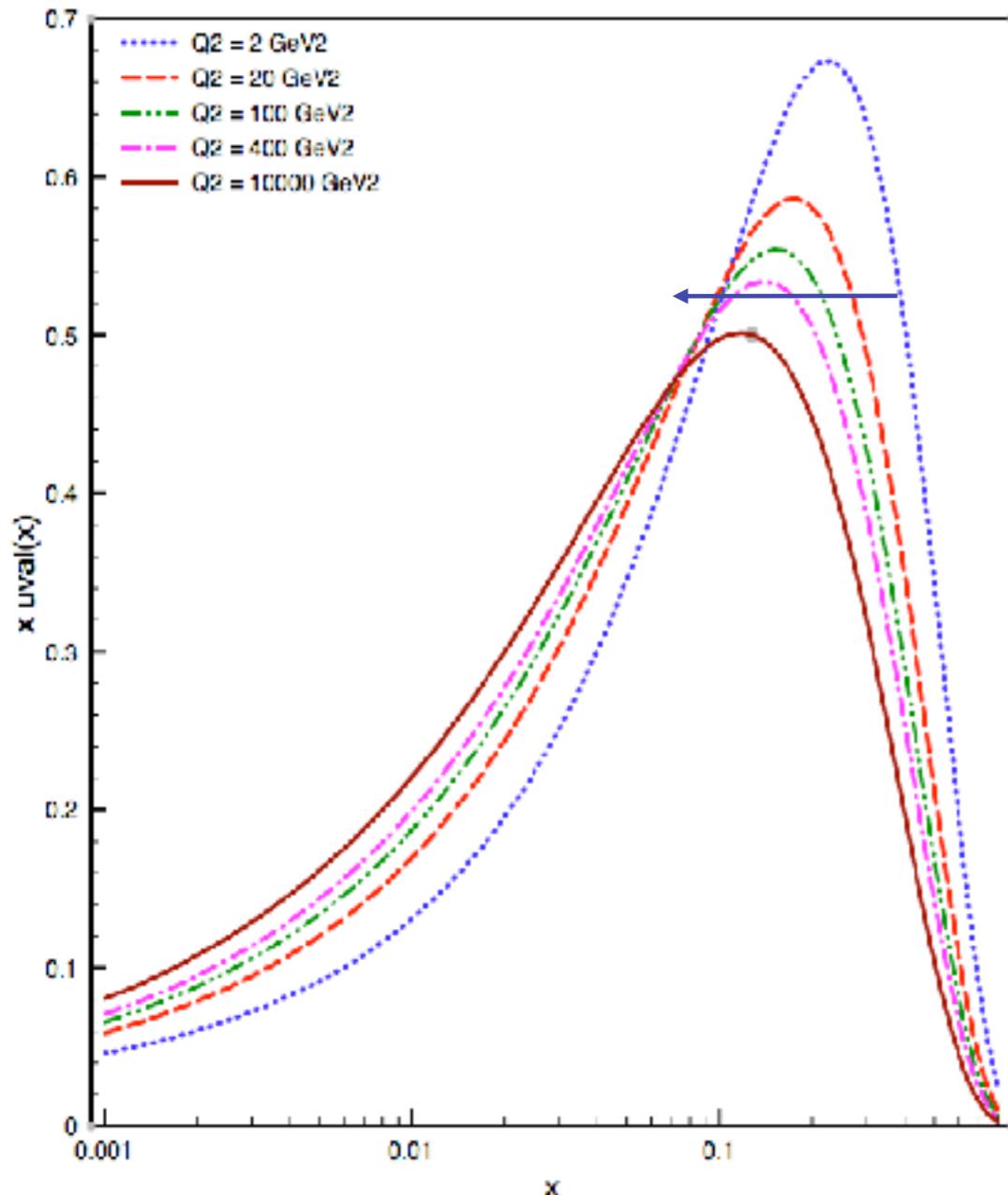
$$\Sigma(x, \mu^2) = \Gamma_{qq} \otimes \Sigma(x, \mu_0^2) + \Gamma_{qg} \otimes g(x, \mu_0^2)$$

$$P_{qq}^{(0)}(x) = C_F \left[ \frac{(1+x^2)}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right]$$

$$P_{qg}^{(0)}(x) = T_R [x^2 + (1-x)^2]$$

- High- $x$  gluon feeds growth of small- $x$  gluon and quark
- Gluons can be seen because they help drive the quark evolution

# DGLAP evolution equations



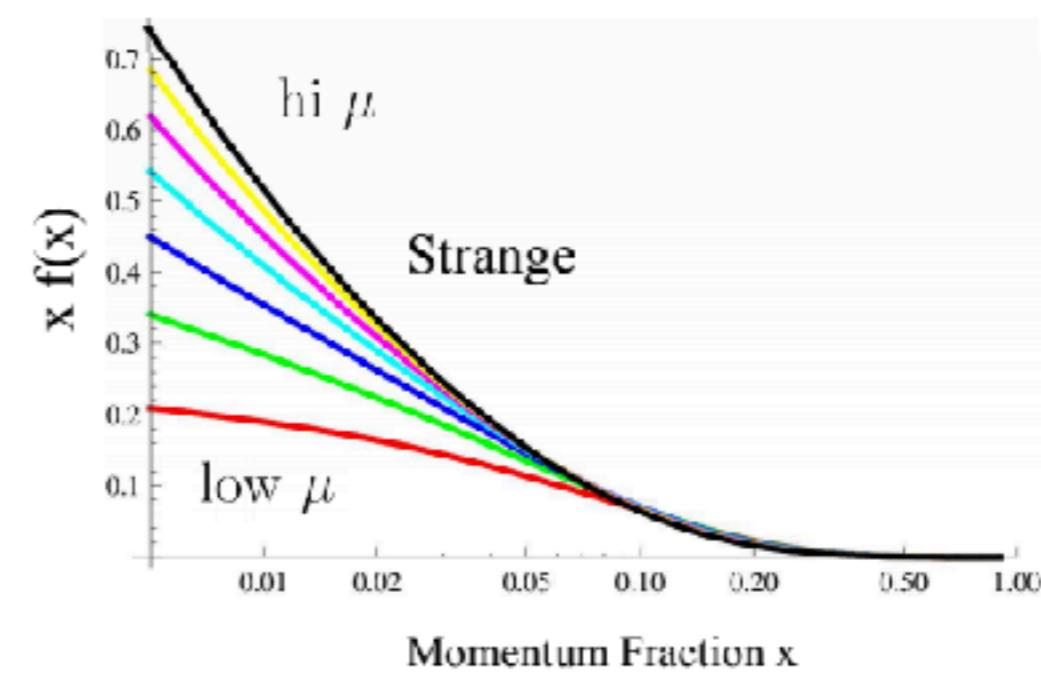
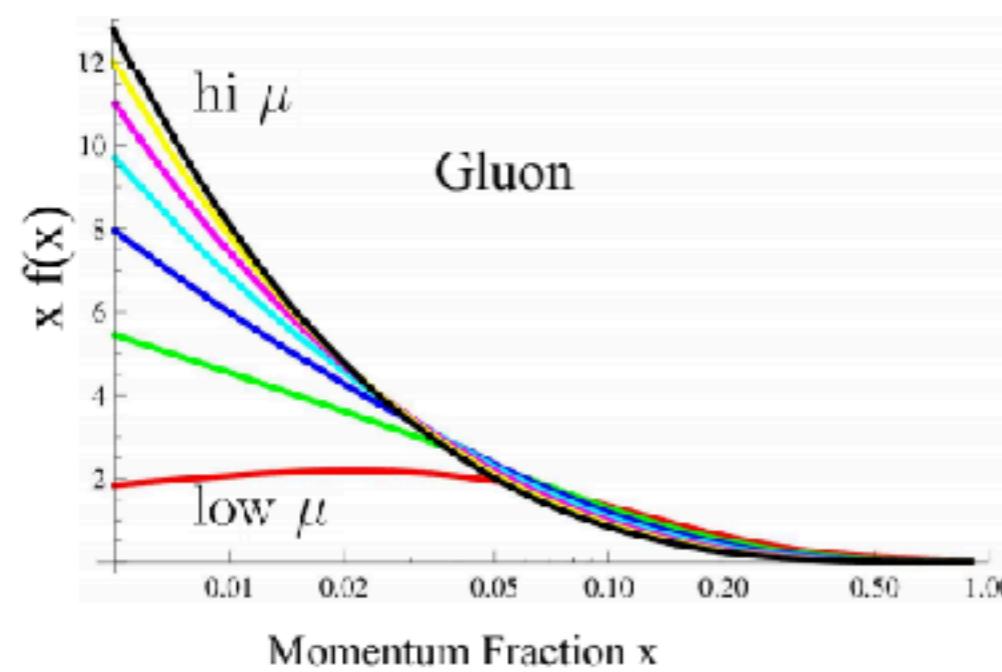
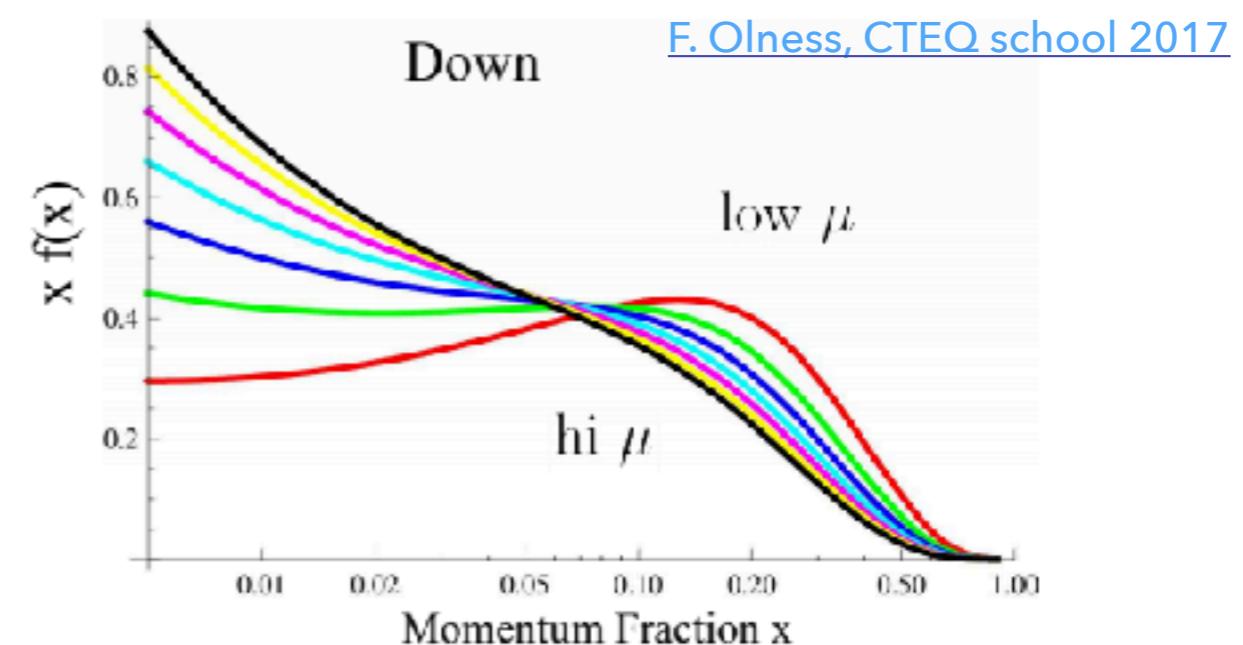
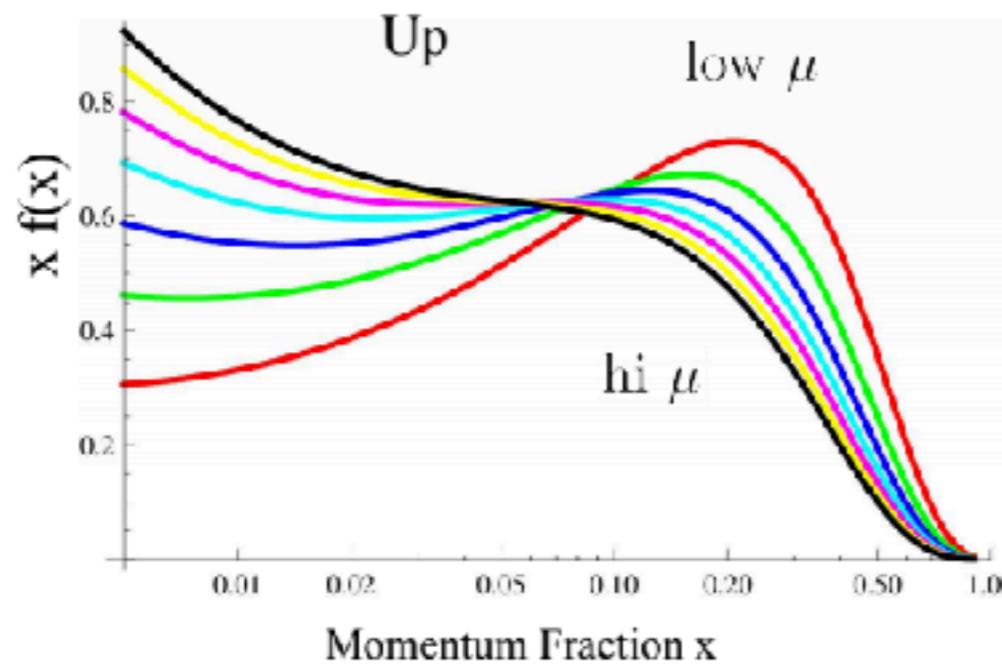
Non-singlet valence evolution

$$u_v(x, \mu^2) = \Gamma_{NS}^v \otimes u_v(x, \mu_0^2)$$

$$P_{NS}^{(0),v} = P_{qq}^{(0)}(x) = C_F \left[ \frac{(1+x^2)}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right]$$

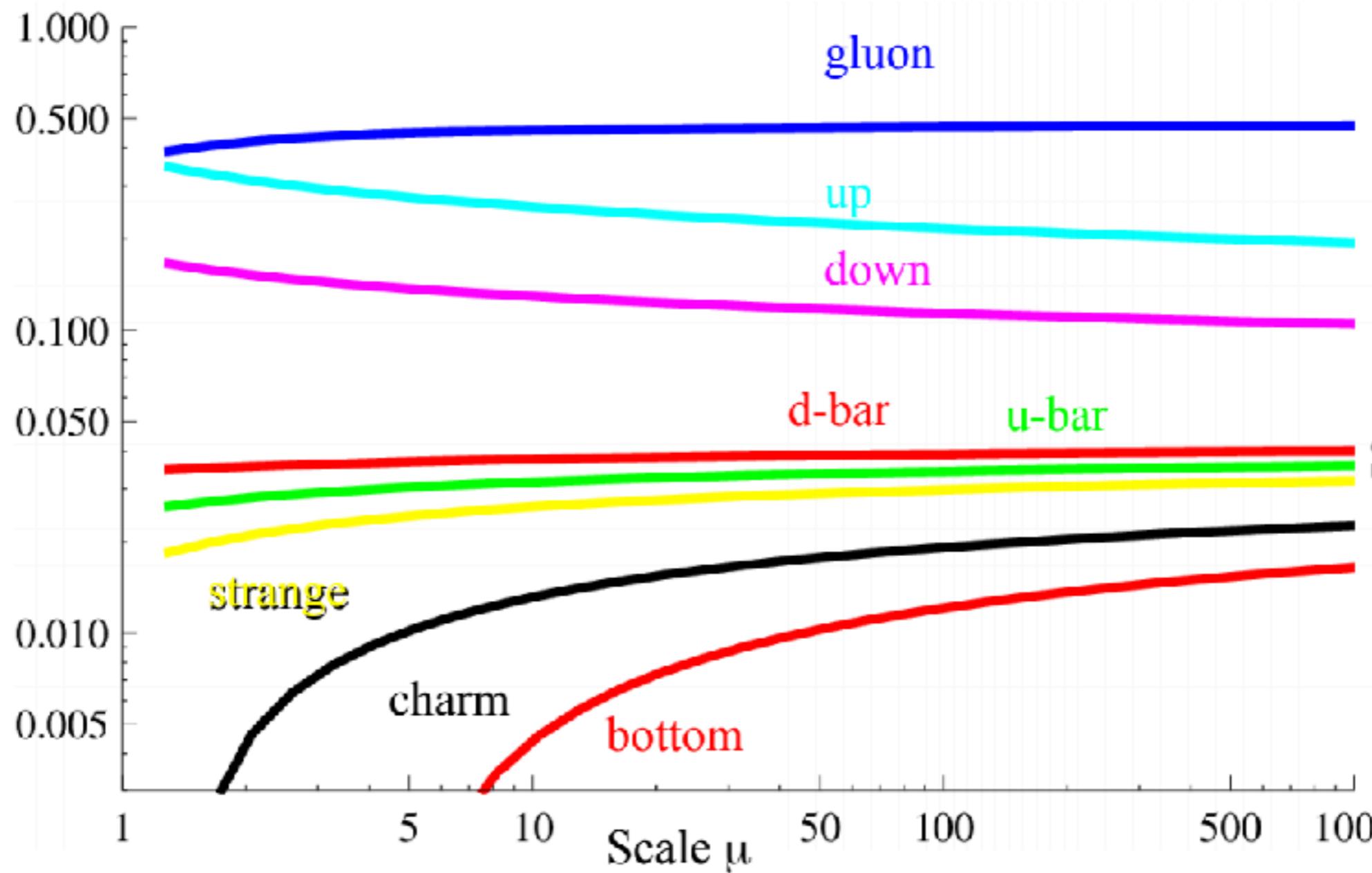
- As  $Q^2$  increases partons lose longitudinal momentum; distributions all shift to lower  $x$
- Gluons can be seen because they help drive the quark evolution

# DGLAP evolution equations



# DGLAP evolution equations

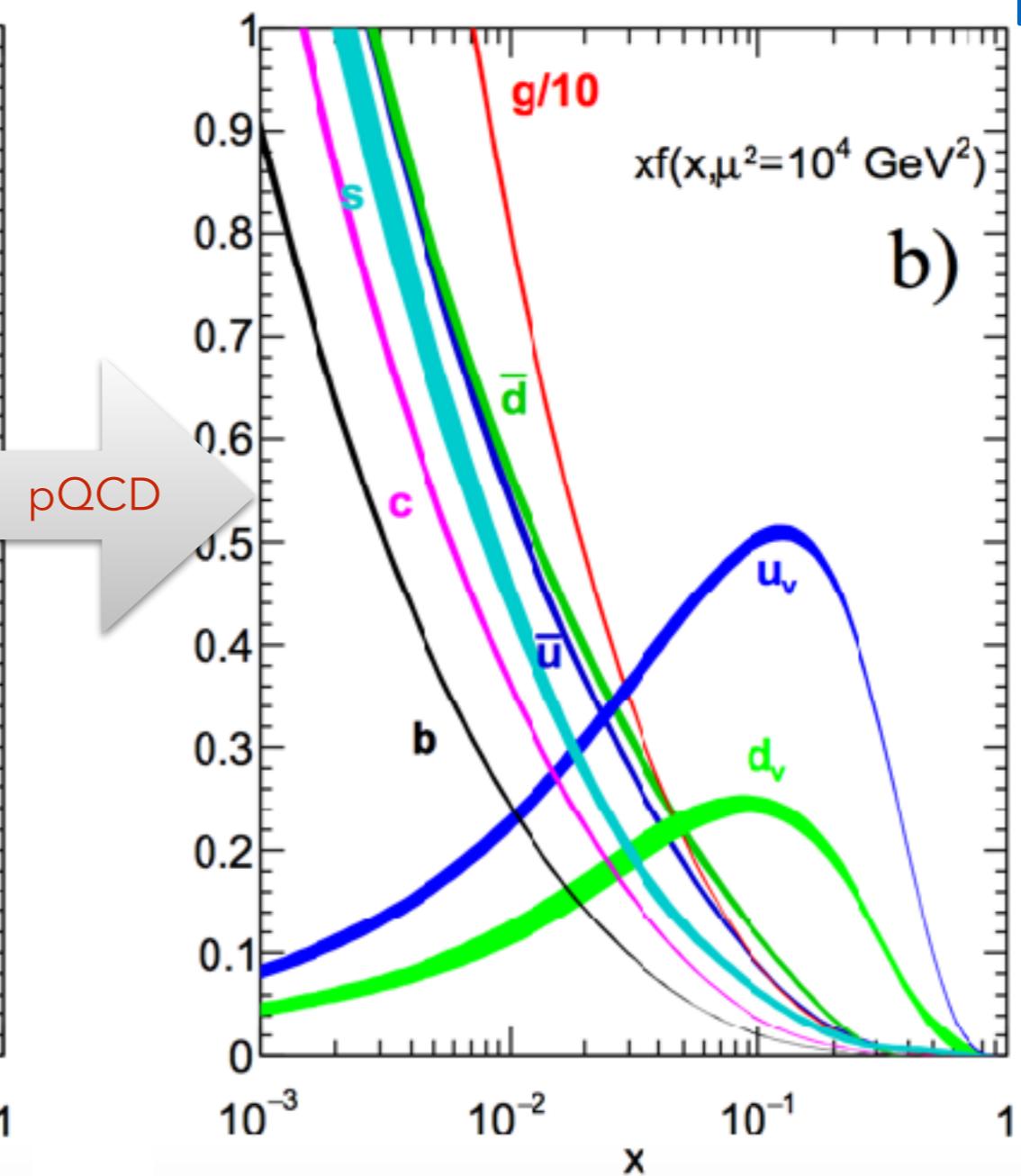
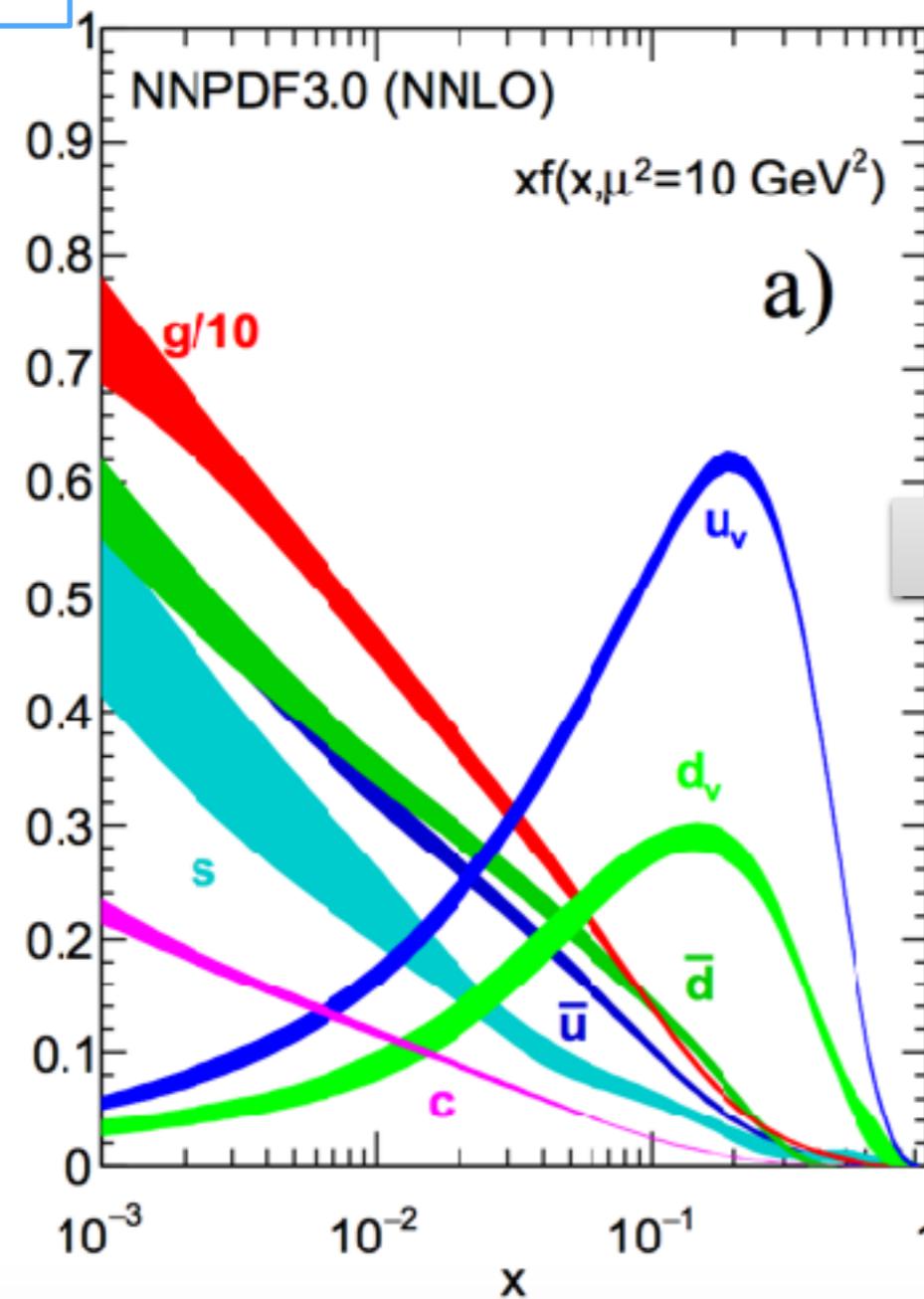
[F. Olness, CTEQ school 2017](#)



# DGLAP evolution equations

Functional dependence of PDFs on the scale is totally predicted up to NNLO accuracy by solving DGLAP evolution equations

Hadronic scale:  
global fit of PDFs



# Wrap-up

- The structure of the proton has been a crucial ingredient to test and verify perturbative QCD and it is now key to the precision challenge that we are facing at the LHC
- Today's lecture
  - ✓ Parametrisation of the proton in terms of structure functions
  - ✓ Parton model picture
  - ✓ QCD - Improved parton model
  - ✓ DGLAP evolution equations
  - ✓ Collinear Factorisation Theorem

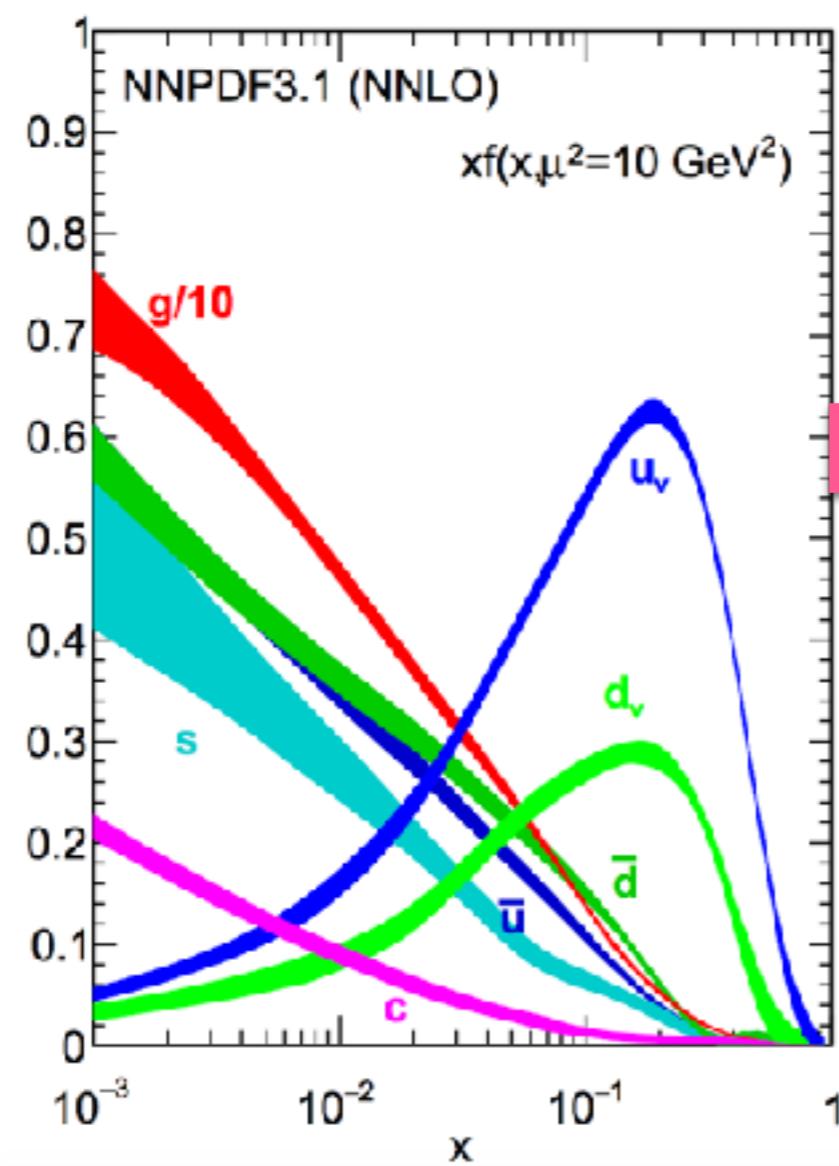
# PDF determination

$f_i(x, \mu)$

Data

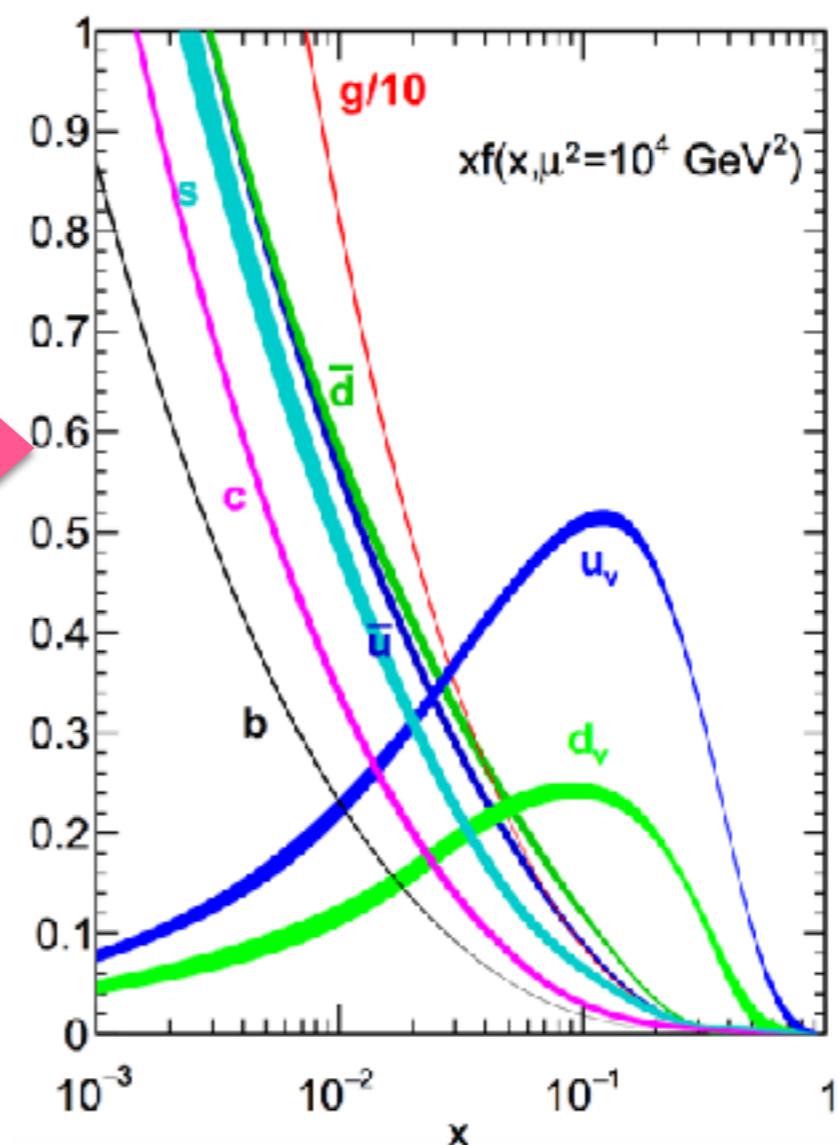
Perturbative QCD

Hadronic scale:  
global fit of PDFs

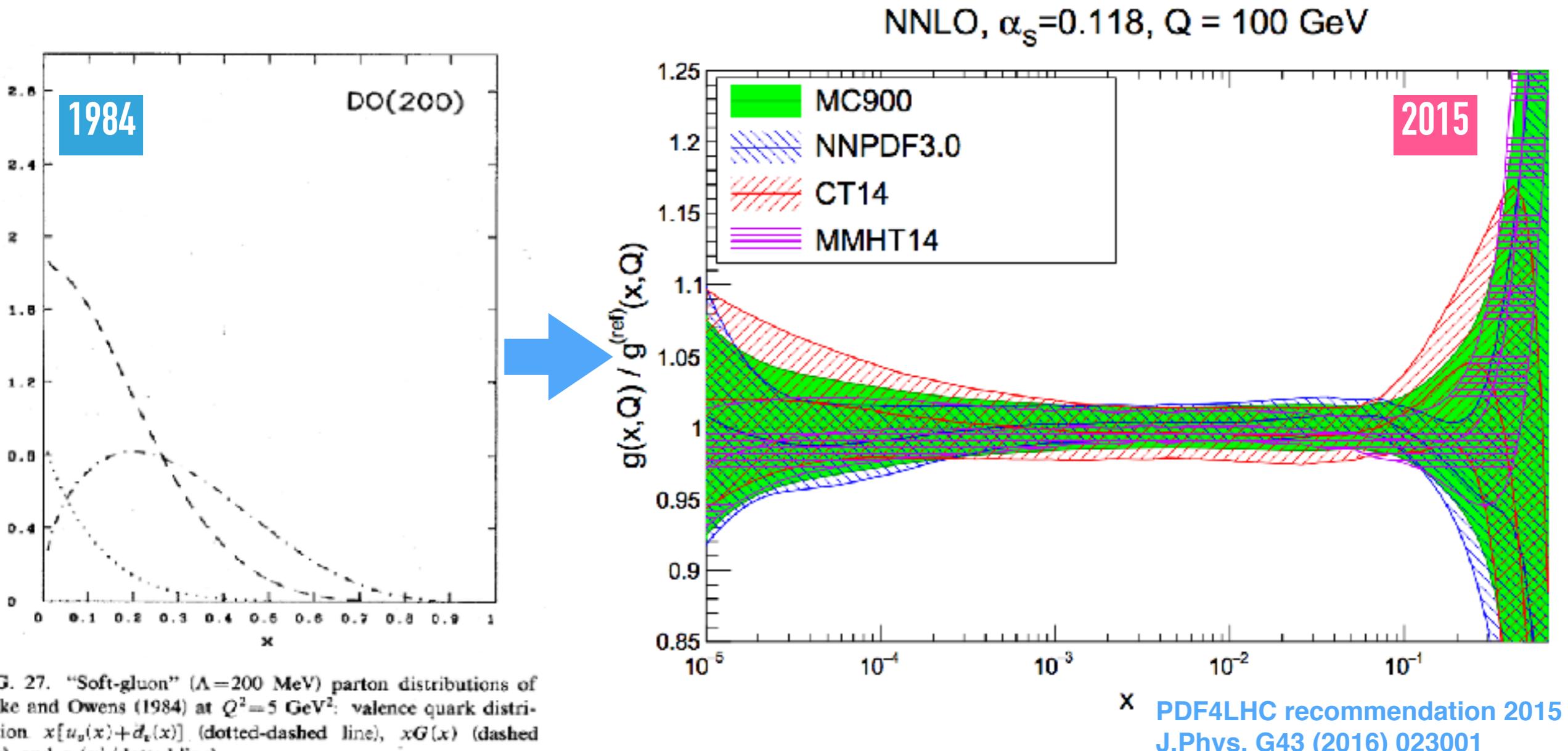


pQCD

High scale:  
input to the LHC



# PDF determination



- ★ 30 years of steady progress in PDF community have produced a huge impact on understanding of proton structure and precision physics

# The ingredients

# The ingredients

- Choose **experimental data** to fit and include all info on correlations
- **Theory settings:** perturbative order, heavy quark mass scheme, EW corrections, intrinsic heavy quarks,  $a_s$ , quark masses value and scheme
- Choose a starting scale  $Q_0$  where pQCD applies
- **Parametrise** independent quarks and gluon distributions at the starting scale
- Solve **DGLAP equations** from initial scale to scales of experimental data and build up observables
- **Fit** PDFs to data
- Provide **error sets** to compute PDF uncertainties

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# The ingredients

$$\sigma_{\mathcal{F}} = \left( \sum_{k=1}^{N_{\text{set}}} \left( \mathcal{F}[\{f^{(k)}\}] - \mathcal{F}[\{f^{(0)}\}] \right)^2 \right)^{1/2}$$

error sets  
mem > 1

central set  
mem = 0

**call InitPDF(mem)**

**call evolvePDF(x,Q,f)**

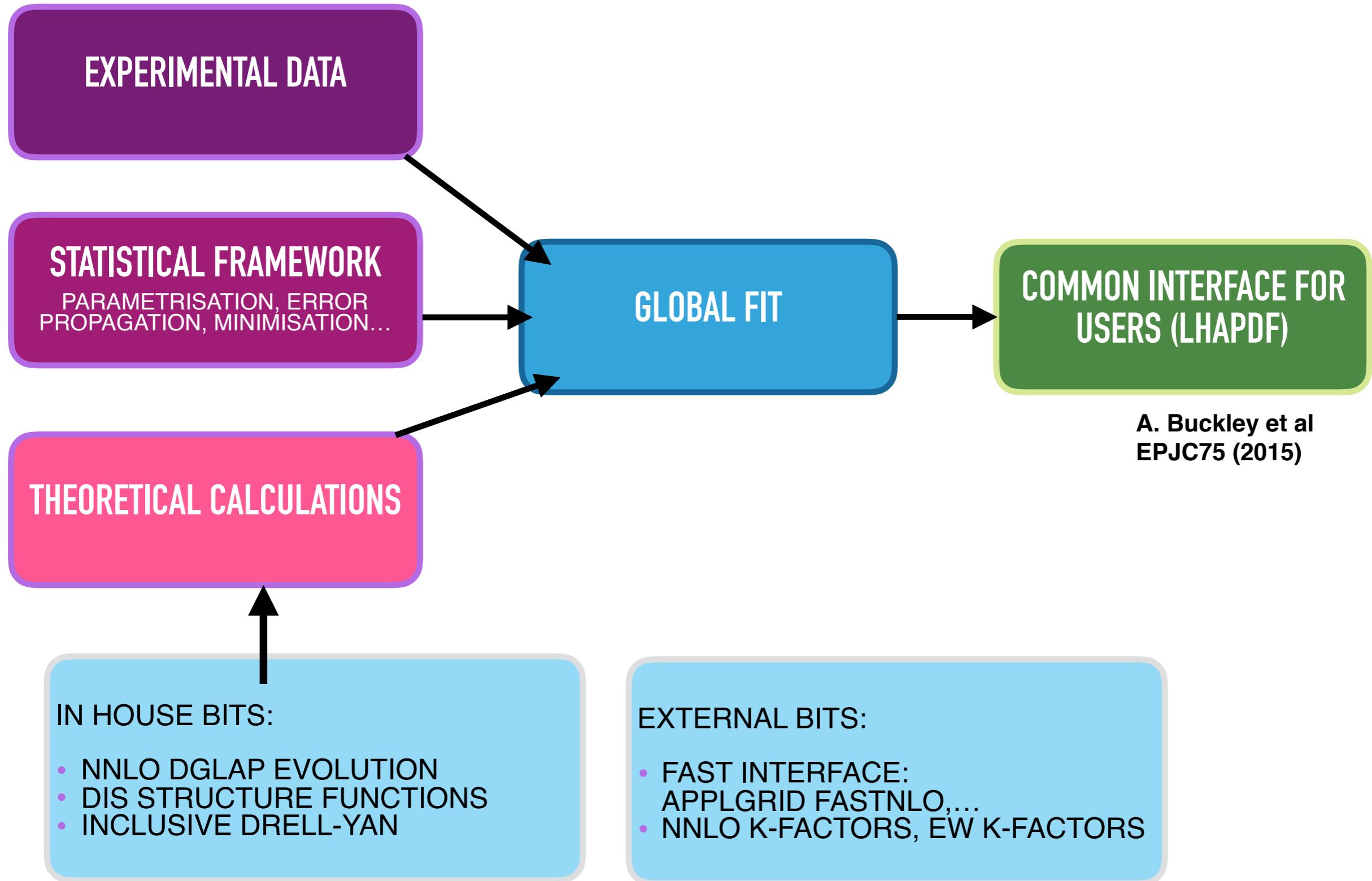
LHAPDF interface

<http://lhapdf.hepforge.org>

- Provide PDF **error sets** to compute PDF uncertainties

	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
Parton	tbar	bbar	cbar	sbar	ubar	dbar	g	d	u	s	c	b	t

# A complex machinery



# Experimental Data

# Experimental data

- PDFs are not measurable, we measure observables that convolute PDFs with partonic cross sections

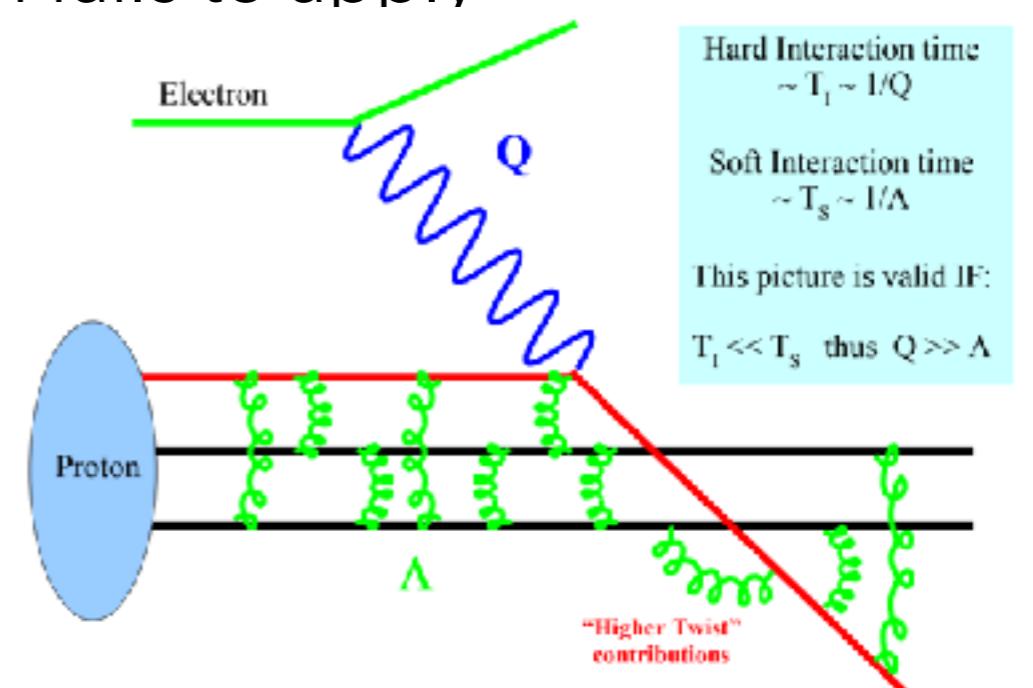
$$\frac{d\sigma_H^{ep \rightarrow ab}}{dX} = \sum_{i=-n_f}^{+n_f} \int_{x_B}^1 \frac{dz}{z} f_i(z, \mu_F) \frac{d\hat{\sigma}_i^{ei}}{dX}(zS, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$

$$\frac{d\sigma_H^{pp \rightarrow ab}}{dX} = \sum_{i,j=-n_f}^{+n_f} \int_{\tau_0}^1 \frac{dz_1}{z_1} \frac{dz_2}{z_2} f_i(z_1, \mu_F) f_j(z_2, \mu_F) \frac{d\hat{\sigma}_i^{ij}}{dX}(zS, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$

- Must exclude regions where factorisation fails to apply (low  $Q^2$  and large  $x$ ). Typically

$$Q_{\min}^2 = 2 \text{ GeV}^2$$

$$W_{\min}^2 = \left( Q^2 \frac{1-x}{x} \right)_{\min} = 12.5 \text{ GeV}^2$$



# Experimental data

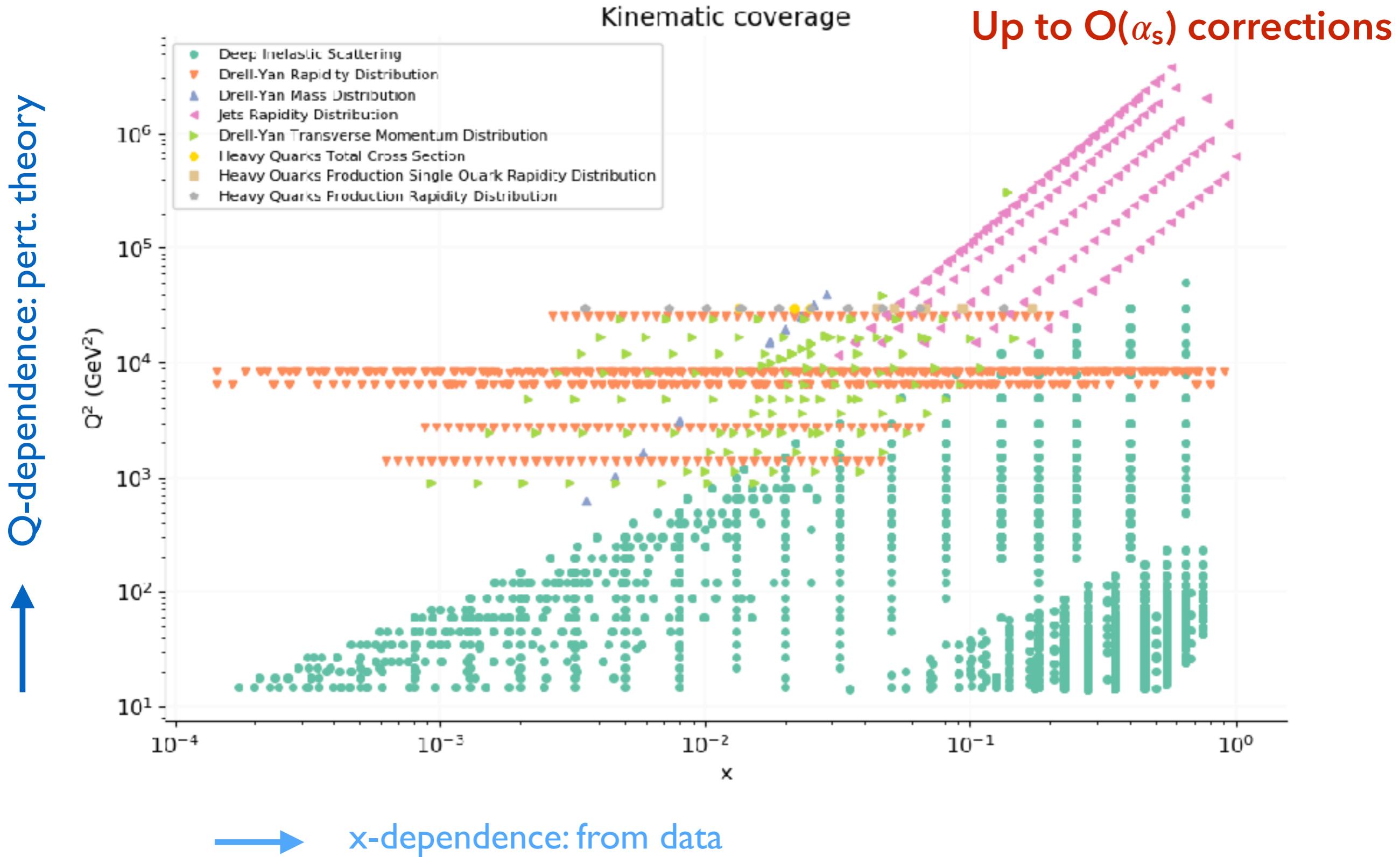
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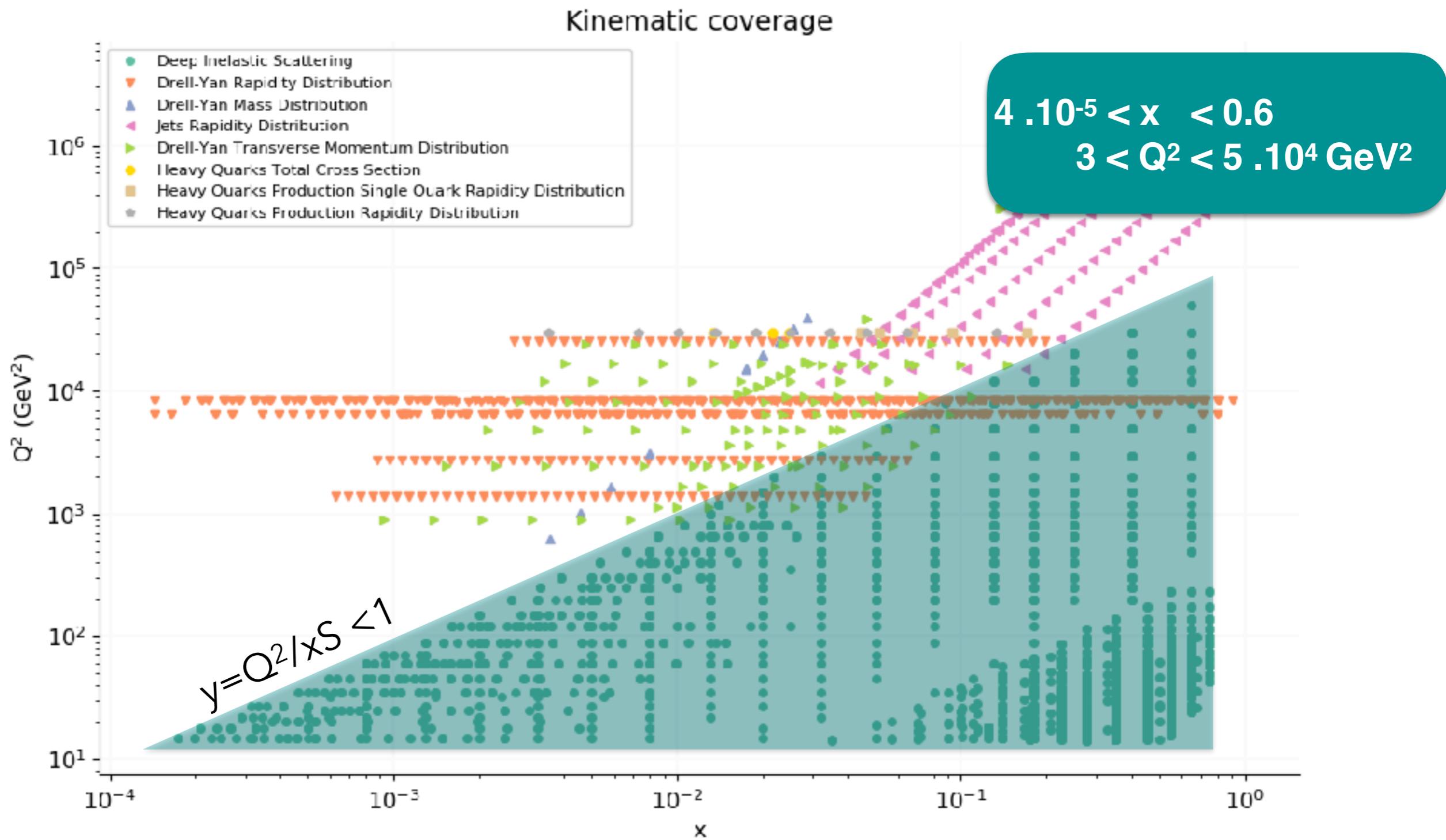
$$\frac{d\sigma_H^{pp \rightarrow ab}}{dX} = \sum_{i,j=-n_f}^{+n_f} \int_{\tau_0}^1 \frac{dz_1}{z_1} \frac{dz_2}{z_2} f_i(z_1, \mu_F) f_j(z_2, \mu_F) \frac{d\hat{\sigma}_i^{ij}}{dX}(zS, \alpha_s(\mu_R), \mu_F) + \mathcal{O}\left(\frac{\Lambda^n}{S^n}\right)$$

- Different data constrain different PDF combinations in different regions
  - DIS data on proton abundant and precise (HERA)
  - In principle  $F_2, F_3$  CC provide 4 light quark combinations
  - $F_2, F_3$  NC provide 2 extra light quark combinations
  - HERA data only determine four combinations of PDFs
  - Old DIS and Drell-Yan data still used because of isospin symmetry
  - $W, Z$  boson final state provide lot of information, gluon from scale dependence
  - Processes with jets and/or heavy quark in final states direct handle on the gluon

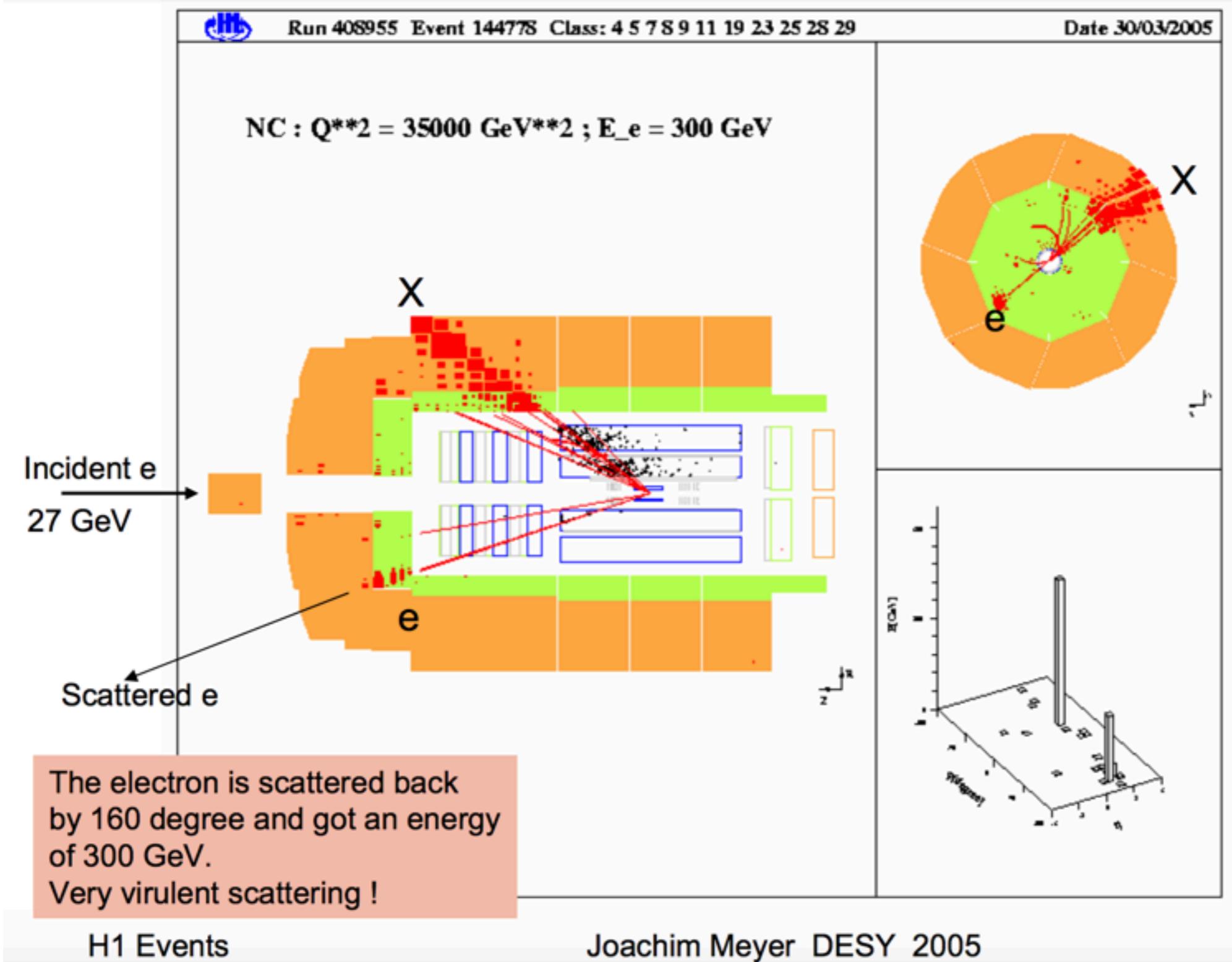
# Disentangling PDFs



# HERA data



# HERA data



Neutral  
Current  
event

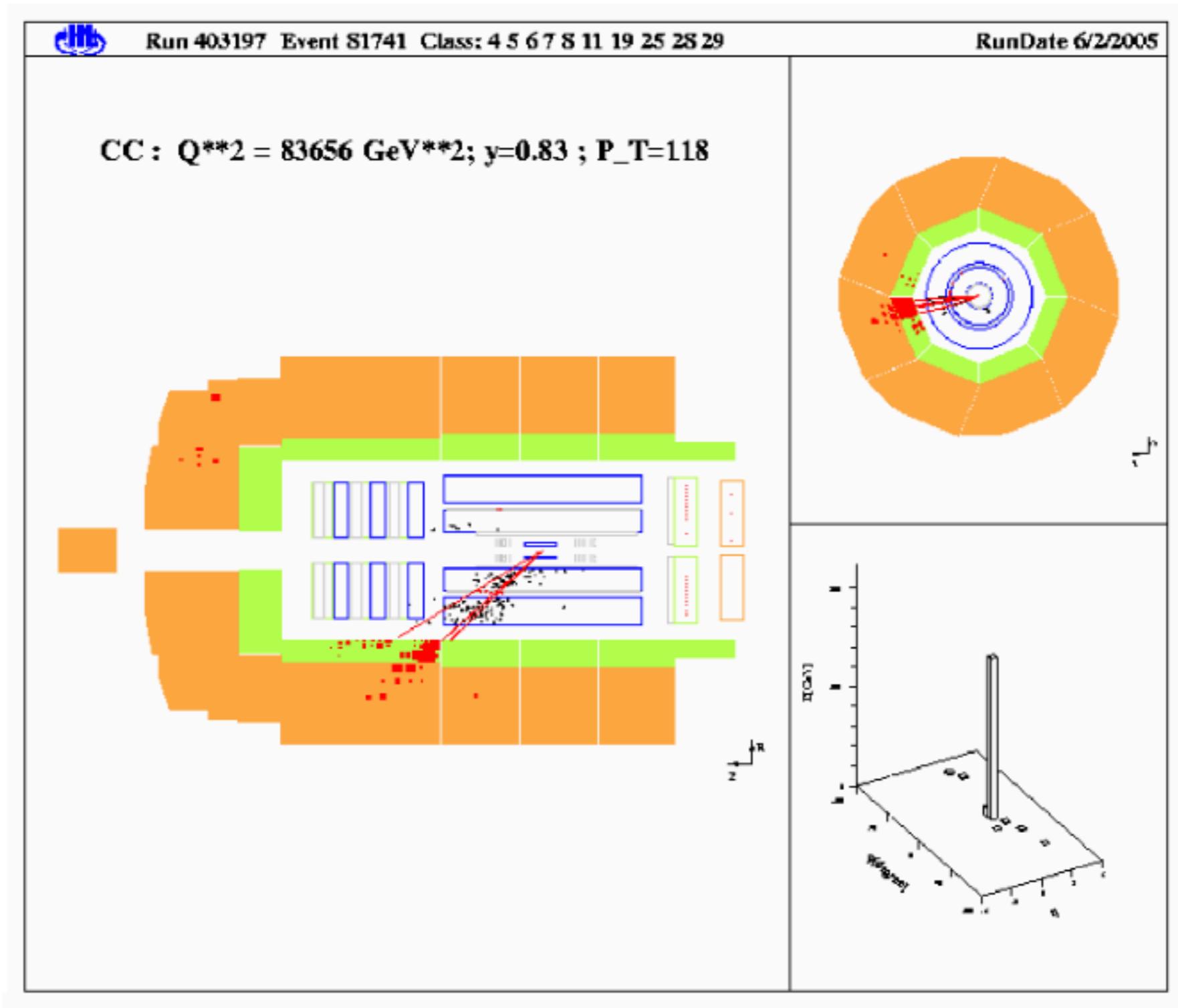
$ep \rightarrow e X$

$Q = 180 \text{ GeV}$

$y = 0.66$

$x_B = 0.47$

# HERA data



Charged  
Current  
event

$ep \rightarrow \nu X$

$Q = 289 \text{ GeV}$

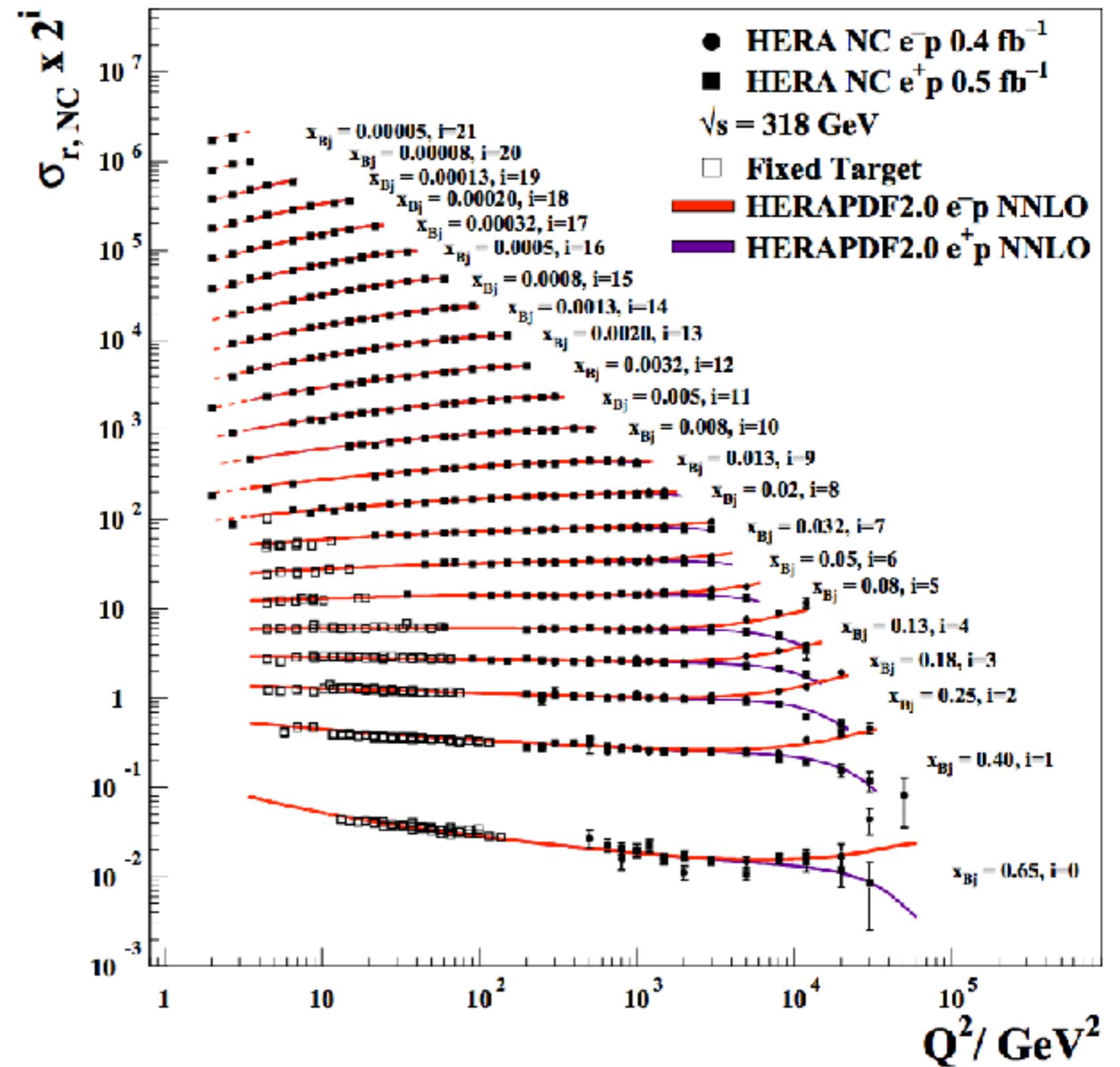
$y = 0.83$

$x_B = 0.91$

# HERA data

## H1 and ZEUS

- Combination of Run I + Run II data led to very precise measurements of reduced xsec
- F3 contribution visible at larger x and  $Q \sim M_Z$



# HERA data

## Neutral Current

$$[F_2^\gamma, F_2^{\gamma Z}, F_2^Z] = x \sum_{i=1}^{n_f} [e_i^2, 2e_i g_V^i, (g_V^i)^2 + (g_A^i)^2] (q_i + \bar{q}_i)$$

③

④

$$[F_3^\gamma, F_3^{\gamma Z}, F_3^Z] = x \sum_{i=1}^{n_f} [0, 2e_i g_A^i, 2g_V^i g_A^i] (q_i - \bar{q}_i)$$

## Charged Current

①	$F_2^{W^-} = 2x(u + \bar{d} + \bar{s} + c),$
②	$F_3^{W^-} = 2x(u - \bar{d} - \bar{s} + c),$
①	$F_2^{W^+} = 2x(d + \bar{u} + \bar{c} + s),$
②	$F_3^{W^+} = 2x(d - \bar{u} - \bar{c} + s),$

$$\frac{d^2\sigma}{dxdQ^2} \propto Y_+ F_2(x, Q^2) \mp Y_- x F_3(x, Q^2) - y^2 F_L(x, Q^2)$$

## Longitudinal Structure function

$$F_L(x, Q^2) = \frac{\alpha_s(Q^2)}{\pi} \left[ \frac{4}{3} \int_x^1 \frac{dy}{y} \left( \frac{x}{y} \right)^2 F_2(y, Q^2) + 2 \sum_i e_i^2 \int_x^1 \frac{dy}{y} \left( \frac{x}{y} \right)^2 \left( 1 - \frac{x}{y} \right) g(y, Q^2) \right]$$

# HERA data

## Neutral Current

$$[F_2^\gamma, F_2^{\gamma Z}, F_2^Z] = x \sum_{i=1}^{n_f} [e_i^2, 2e_i g_V^i, (g_V^i)^2 + (g_A^i)^2] (q_i + \bar{q}_i)$$

③

④

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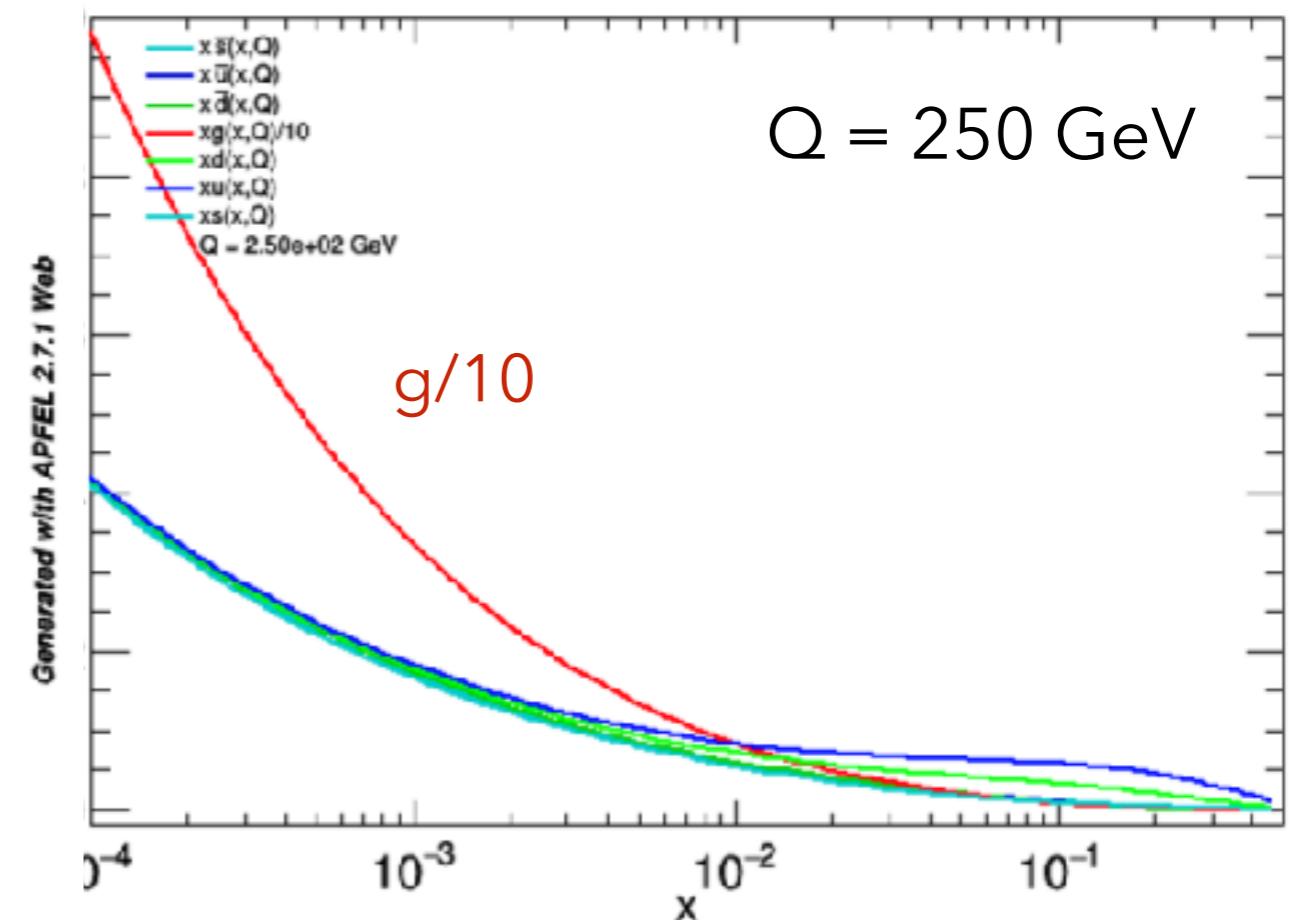
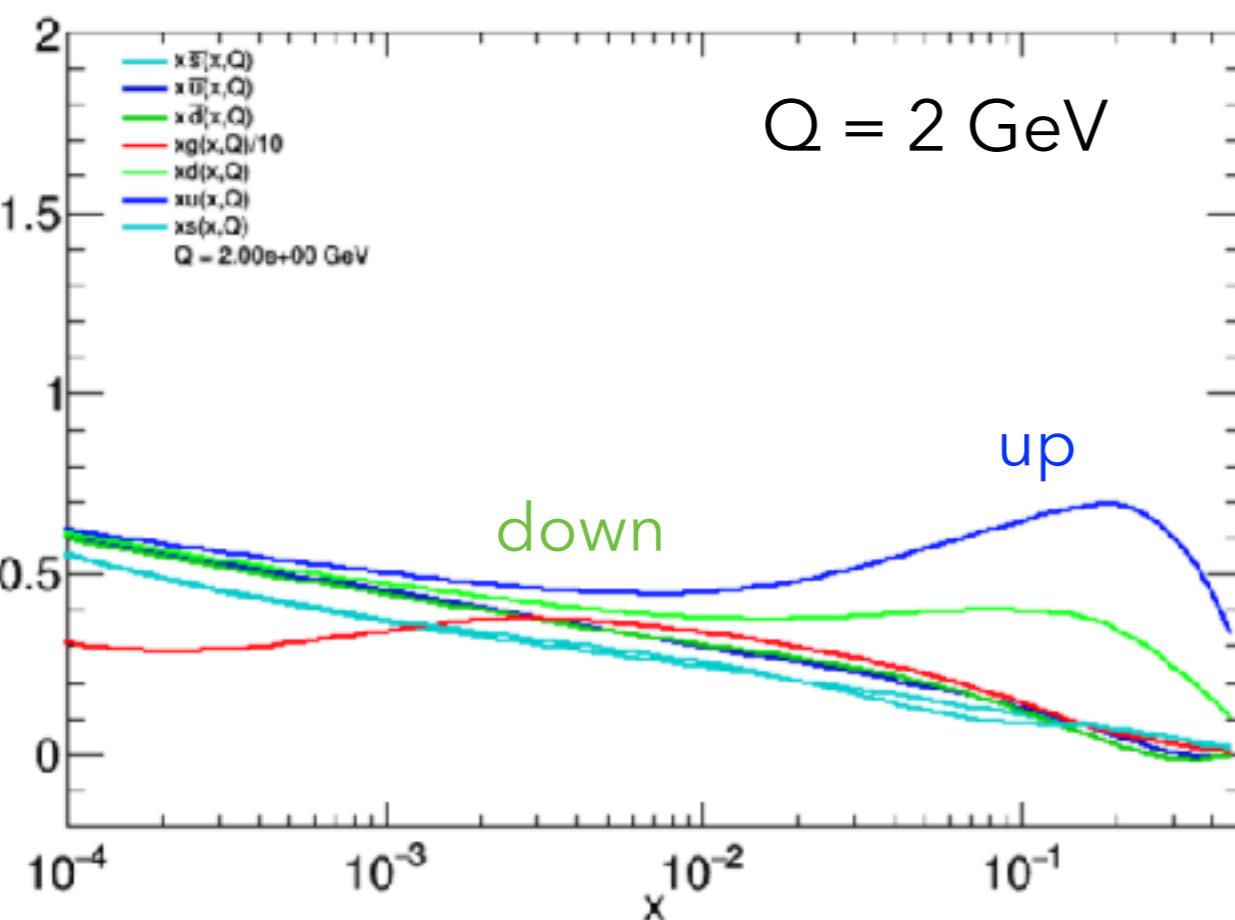
## Charged Current

$$\textcircled{1} \quad F_2^{W^-} = 2x(u + \bar{d} + \bar{s} + c),$$

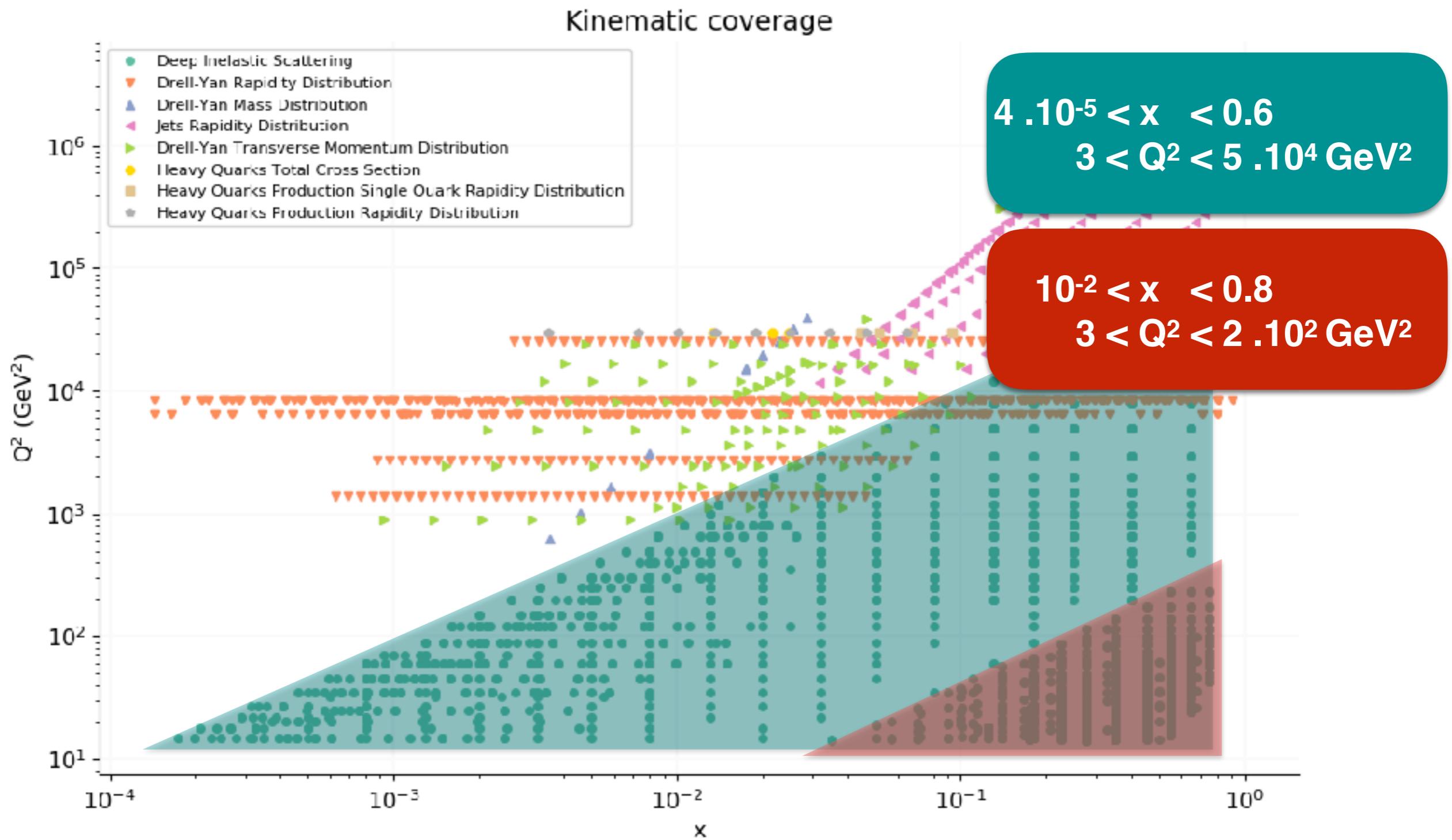
$$F_3^{W^-} = 2x(u - \bar{d} - \bar{s} + c),$$

$$\textcircled{2} \quad F_2^{W^+} = 2x(d + \bar{u} + \bar{c} + s),$$

$$F_3^{W^+} = 2x(d - \bar{u} - \bar{c} + s),$$



# Fixed target DIS data



# Fixed target DIS data

- Assumption (SU(2) isospin): neutron is just like proton with  $u \leftrightarrow d$   
proton = uud  
neutron = ddu  
 $\Rightarrow \mathbf{u}_n(x) = \mathbf{d}_p(x)$  and  $\mathbf{d}_n(x) = \mathbf{u}_p(x)$
- Linear combinations of  $F_2^p$  and  $F_2^n$  give separately  $u_p(x) \equiv u(x)$  and  $d_p(x) \equiv d(x)$ ,
- Experimentally measured is deuteron structure function

$$F_2^d = (F_2^p + F_2^n)/2$$

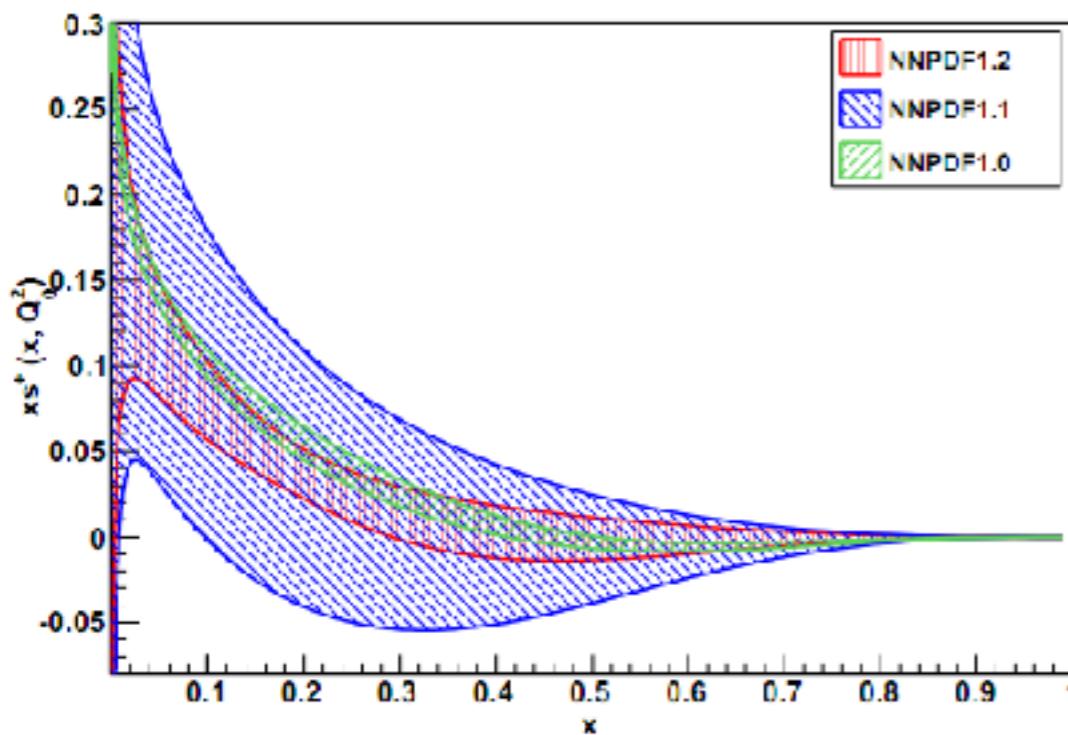
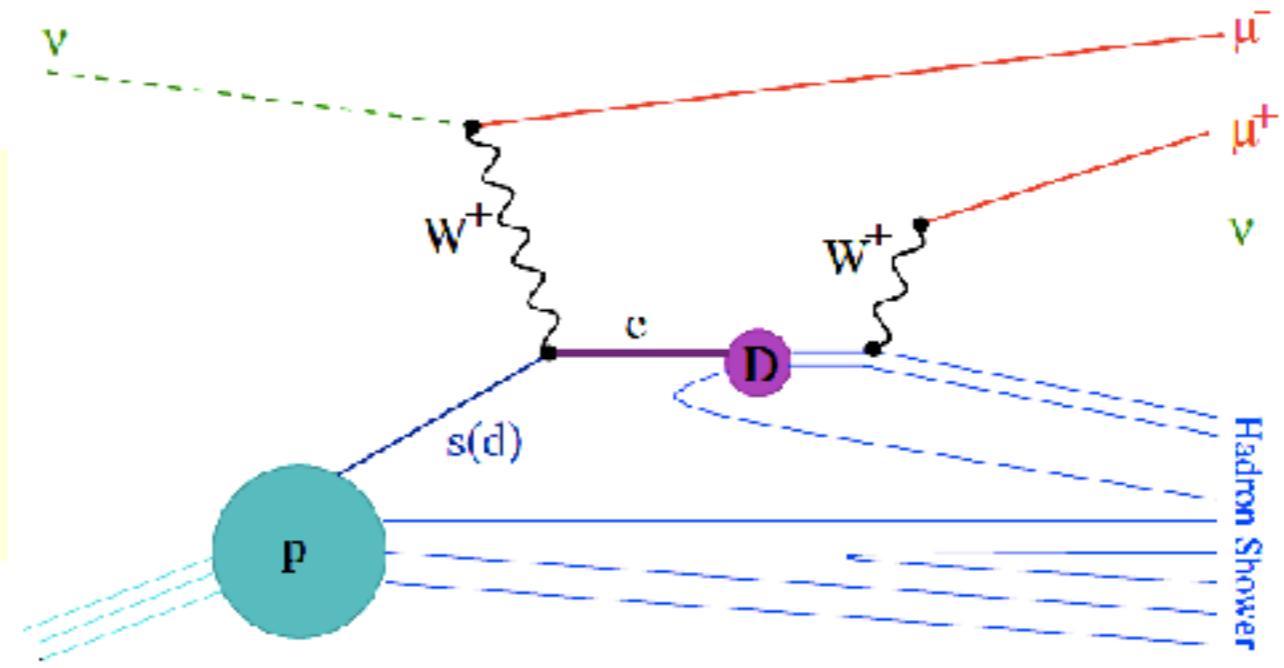
$$F_2^p(x, Q^2) - F_2^d(x, Q^2) = \frac{1}{3}(u + \bar{u} - d - \bar{d})$$

$$\frac{F_2^d(x)}{F_2^p(x)} \sim \frac{u}{d}$$

# Fixed target DIS neutrino

$$\begin{aligned}\tilde{\sigma}^{\nu(\bar{\nu}),c} &\propto (F_2^{\nu(\bar{\nu}),c}, F_3^{\nu(\bar{\nu}),c}, F_L^{\nu(\bar{\nu}),c}) \\ F_2^{\nu,c} &= \times \left[ C_{2,q} \otimes 2|V_{cs}|^2 s + \frac{1}{n_f} C_{2,g} \otimes g \right] \\ F_2^{\bar{\nu},c} &= \times \left[ C_{2,q} \otimes 2|V_{cs}|^2 \bar{s} + \frac{1}{n_f} C_{2,g} \otimes g \right]\end{aligned}$$

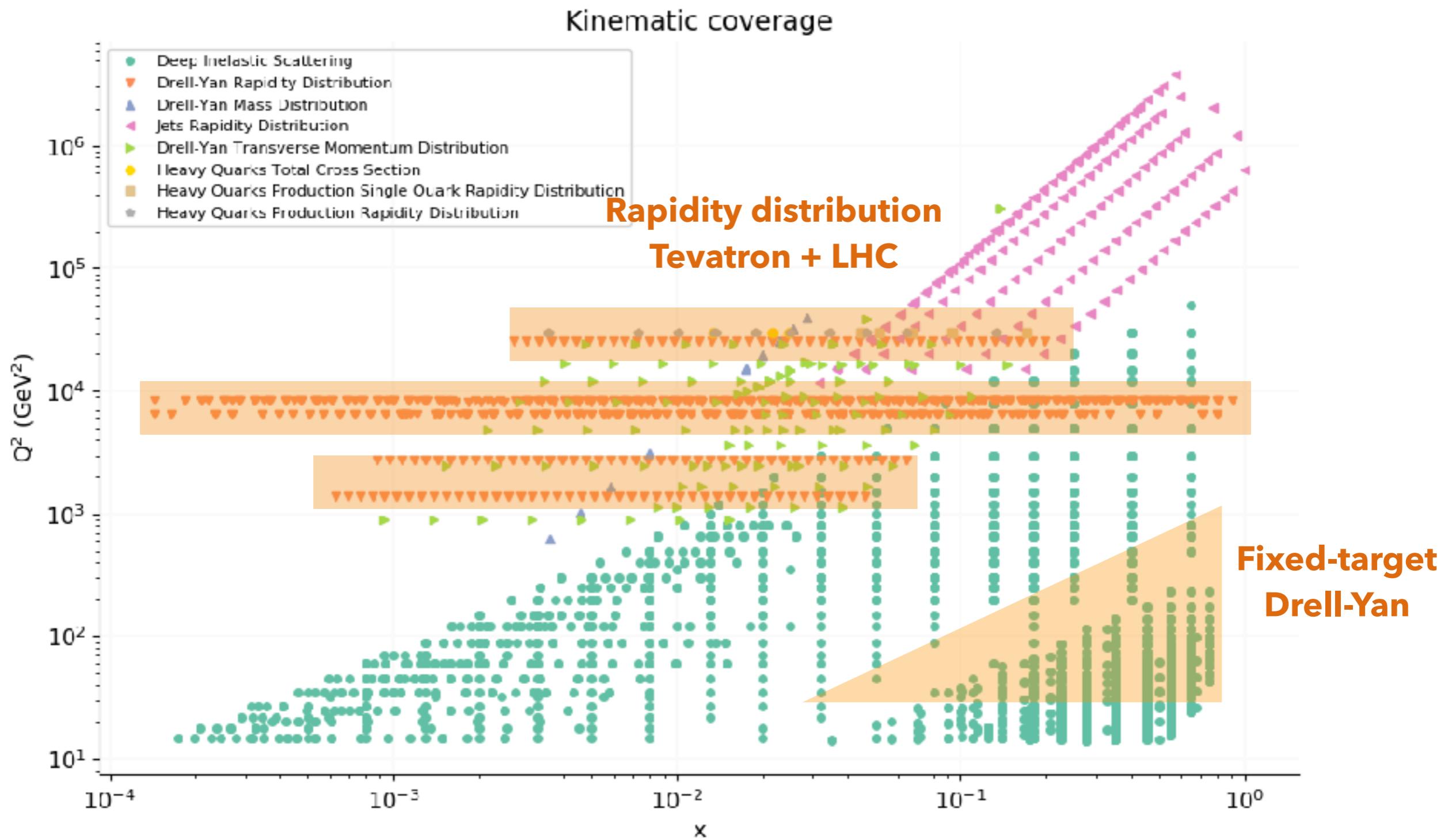
V<sub>cs</sub> enhancement



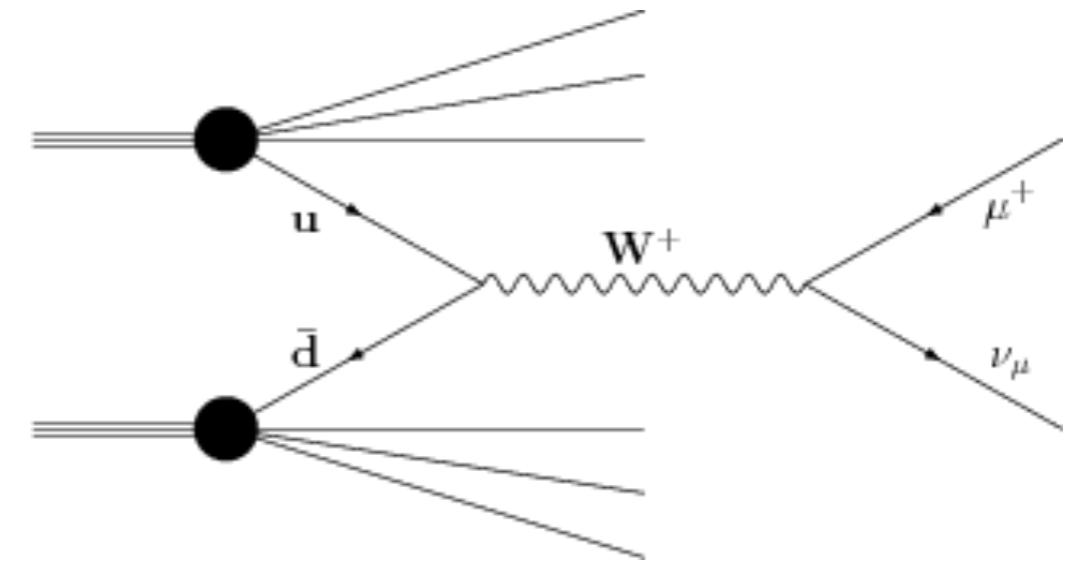
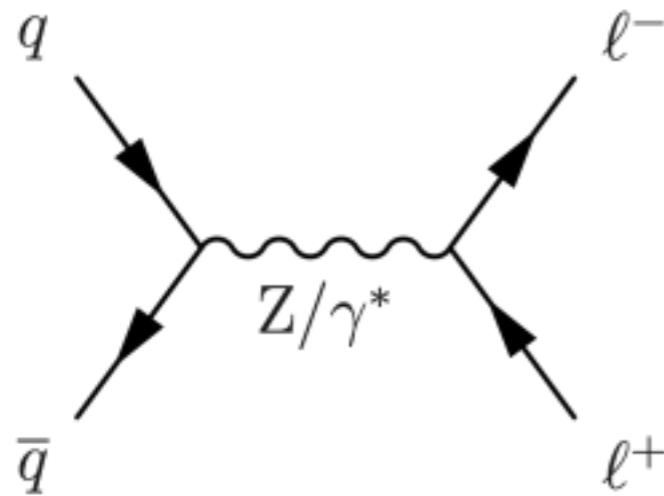
$$x = \frac{Q^2}{2M_n E_\nu y}$$

- Old NuTeV data still provide main constraints on strangeness inside the proton
- Some (mild) tension between fixed target data and W+c data at the LHC

# Drell-Yan/ $\bar{V}$ production data



# Drell-Yan/V production data



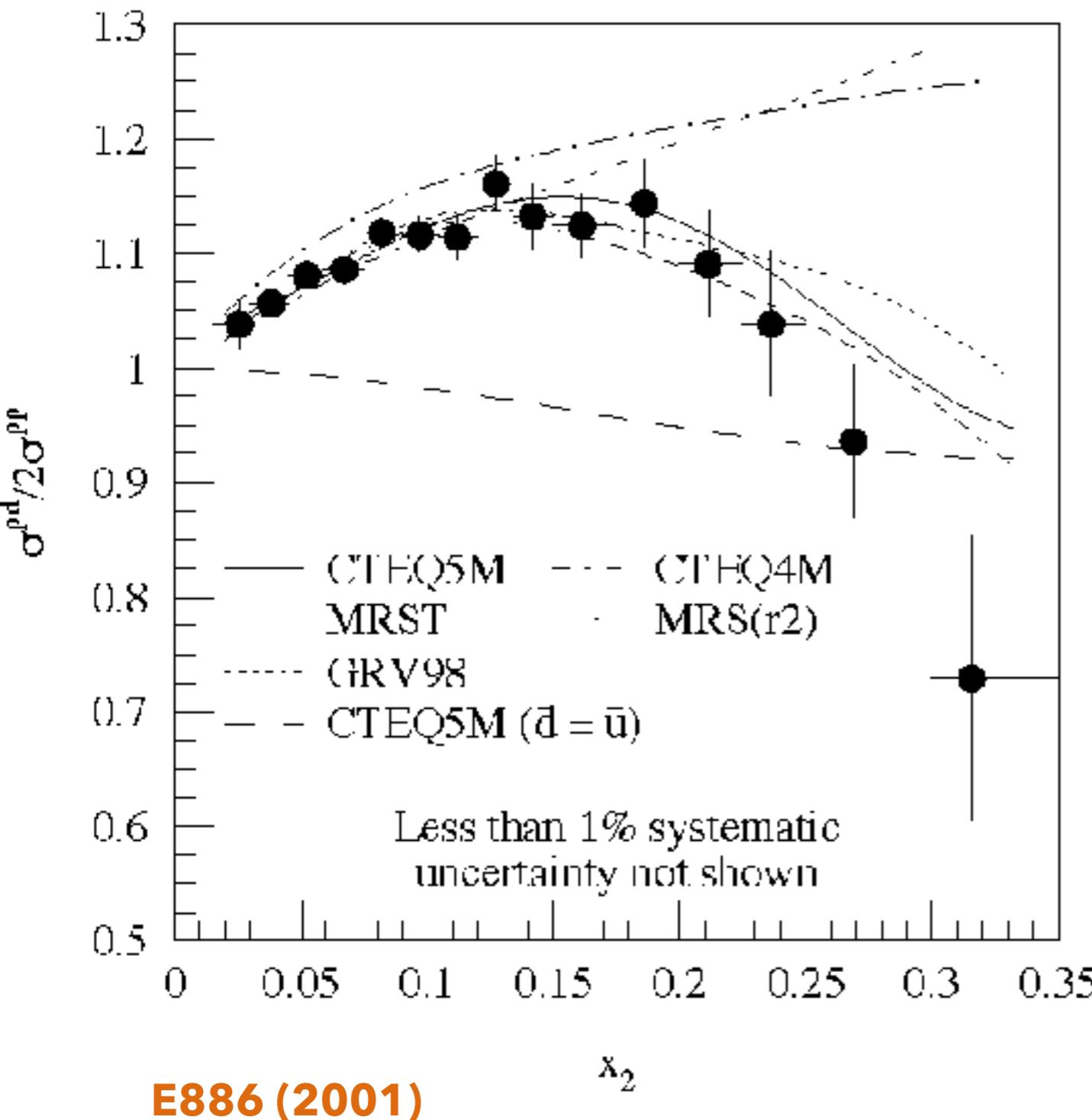
$$L_{ij}(x_1, x_2) = q_i(x_1)\bar{q}_j(x_2)$$

$$\gamma^* : \frac{d\sigma}{dy dM^2} = \frac{4\pi\alpha^2}{9M^2S} \sum_i e_i^2 L_{ij}(x_1, x_2)$$

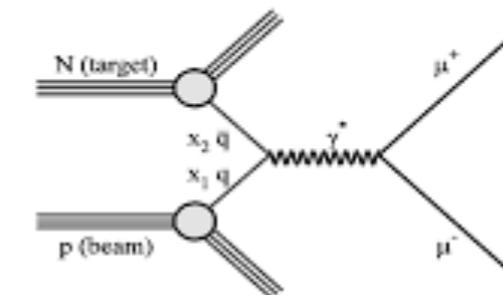
$$Z : \frac{d\sigma}{dy} = \frac{\pi G_F M_V^2 \sqrt{2}}{3S} \sum_i (v_{iZ}^2 + a_{iZ}^2) L_{ij}(x_1, x_2)$$

$$W : \frac{d\sigma}{dy dM^2} = \frac{\pi G_F M_V^2 \sqrt{2}}{3S} \sum_{ij} |V_{ij}^{\text{CKM}}|^2 L_{ij}(x_1, x_2)$$

# Drell-Yan data

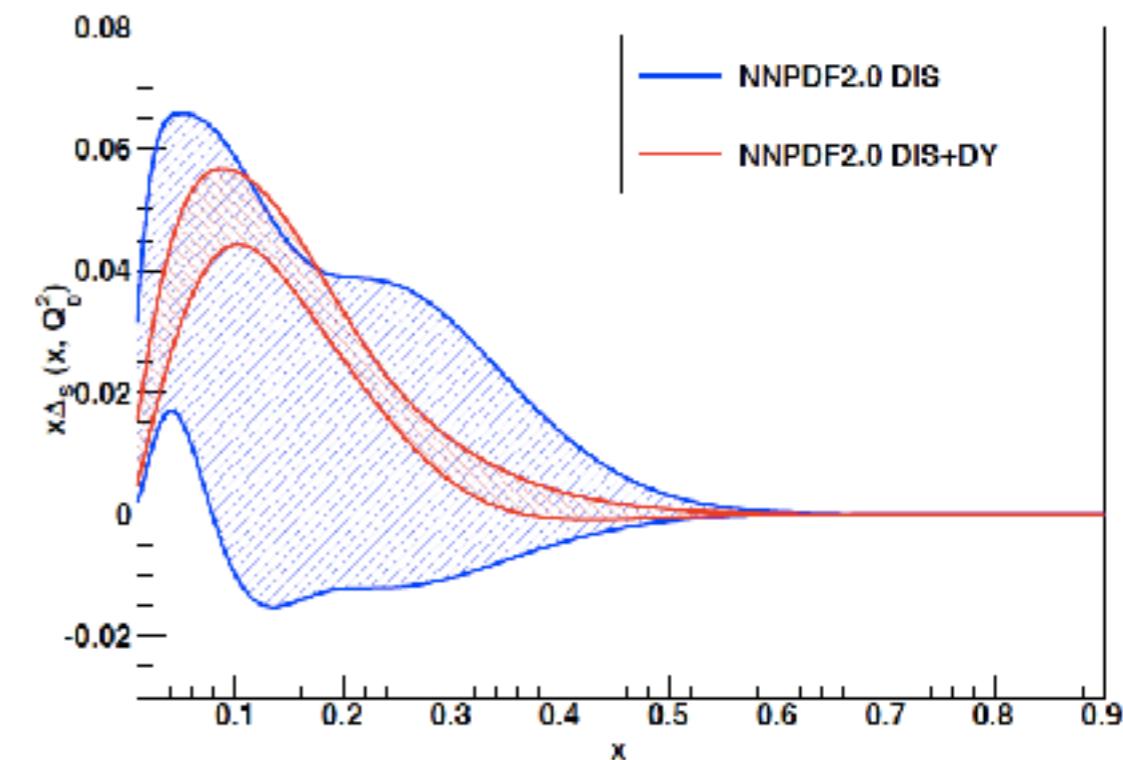


The Drell-Yan Process:  $pN \rightarrow \mu^+ \mu^- X$

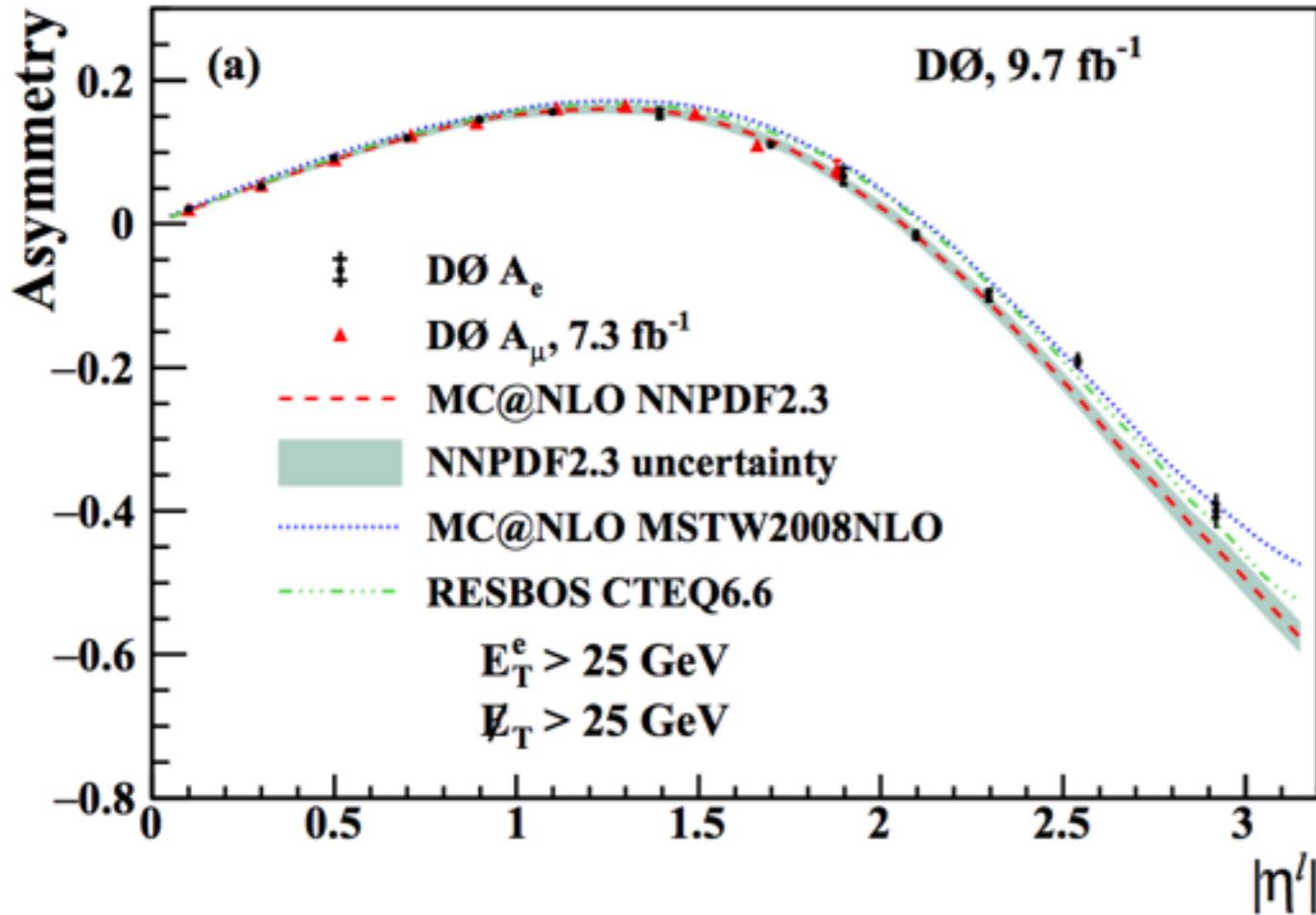


**Fixed-target  
Drell-Yan**

$$\frac{\sigma(pd \rightarrow \mu^+ \mu^-)}{\sigma(pp \rightarrow \mu^+ \mu^-)} = \frac{\frac{4}{9}u\bar{d} + \frac{1}{9}d\bar{u}}{\frac{4}{9}u\bar{u} + \frac{1}{9}d\bar{d}} \sim \frac{\bar{d}}{\bar{u}}$$



# Z/W production data



**W asymmetry at Tevatron**

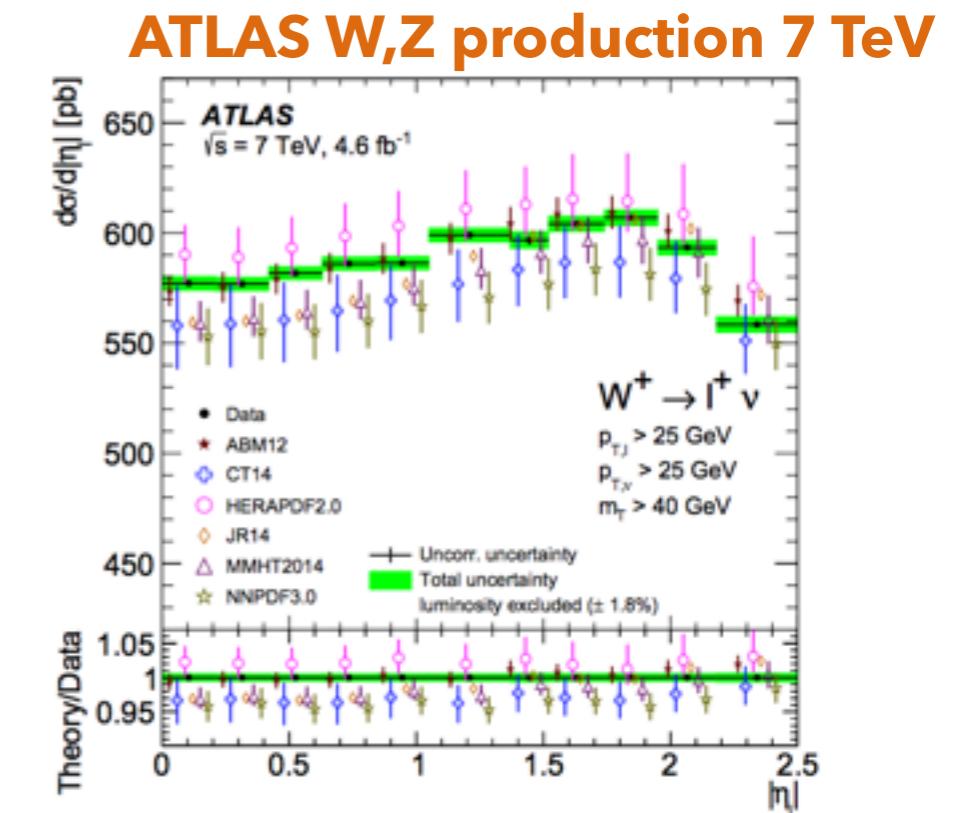
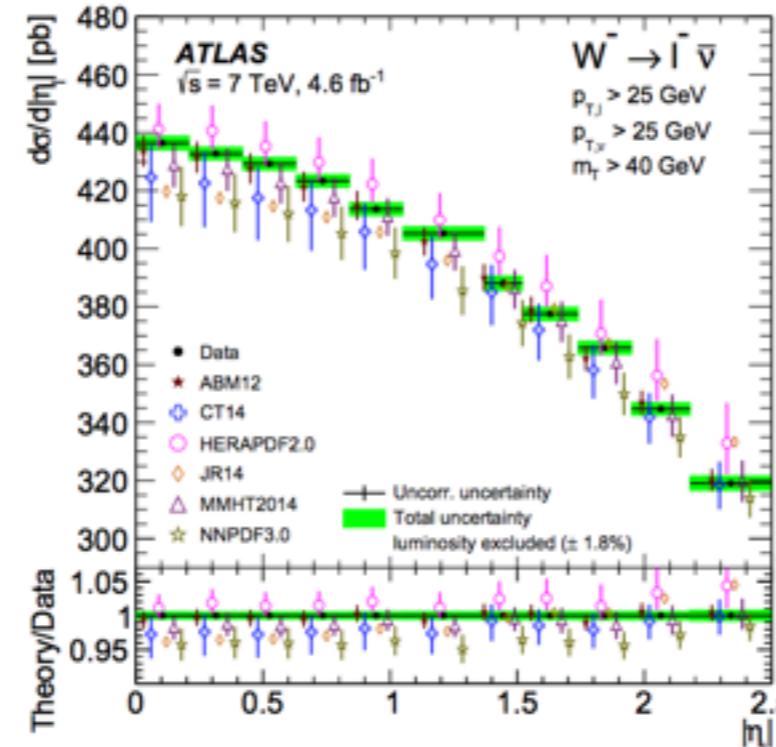
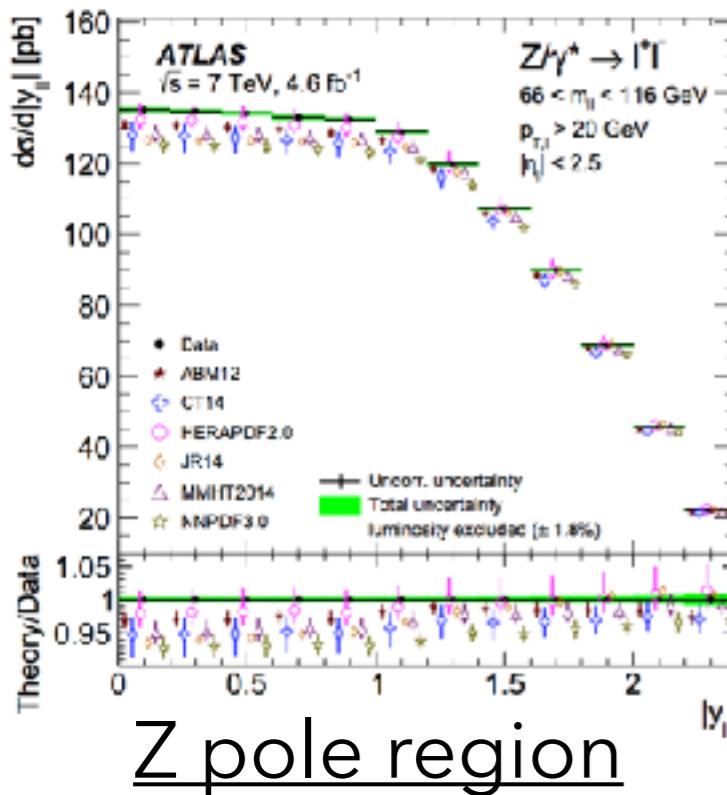
$$u^{\bar{p}} = \bar{u}^p$$

$$d^{\bar{p}} = \bar{d}^p$$

Charge conjugation

$$\frac{\sigma(p\bar{p} \rightarrow W^+)}{\sigma(p\bar{p} \rightarrow W^-)} = \frac{u(x_1)d(x_2) + \bar{u}(x_1)\bar{d}(x_2)}{d(x_1)u(x_2) + \bar{d}(x_1)\bar{u}(x_2)} \sim \frac{u}{d}(x_1) \frac{u}{d}(x_2)$$

# Z/W production data

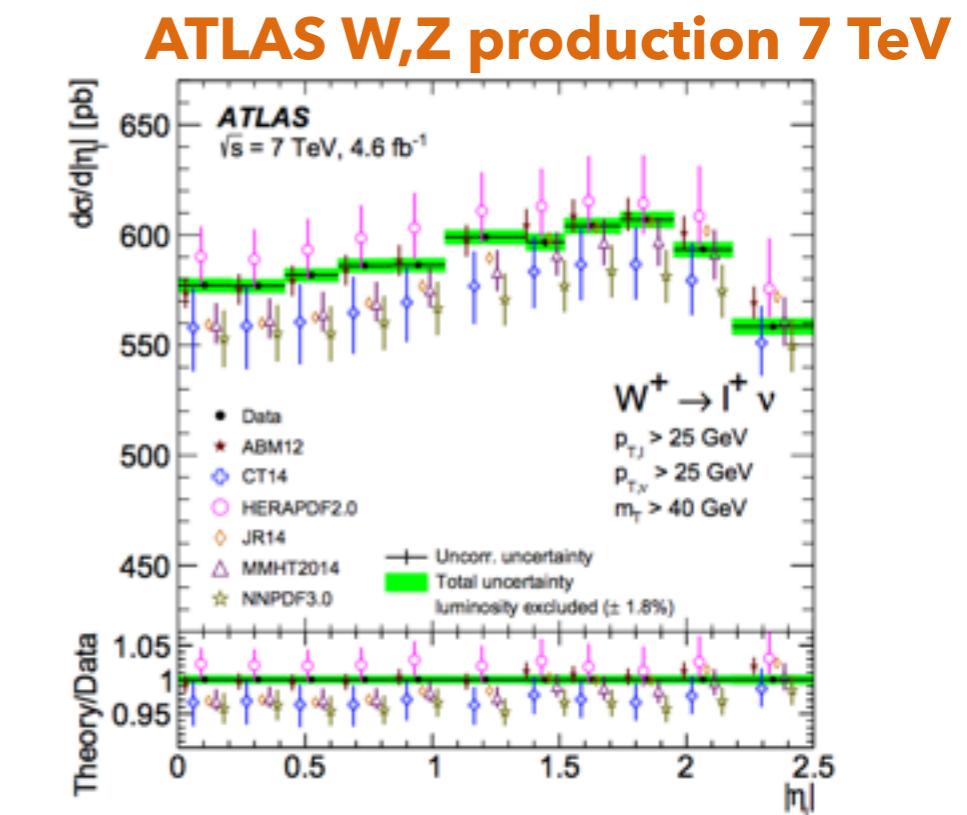
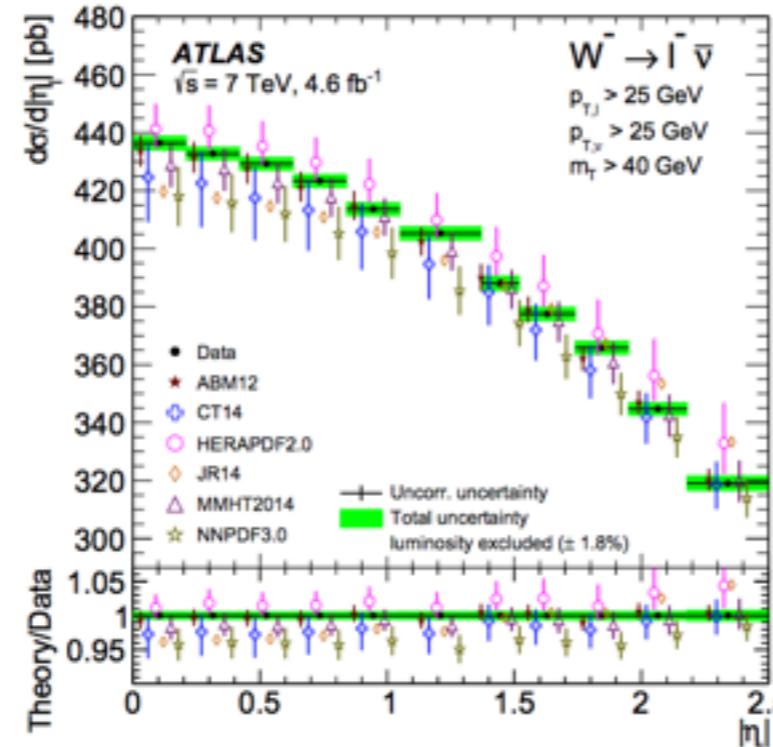
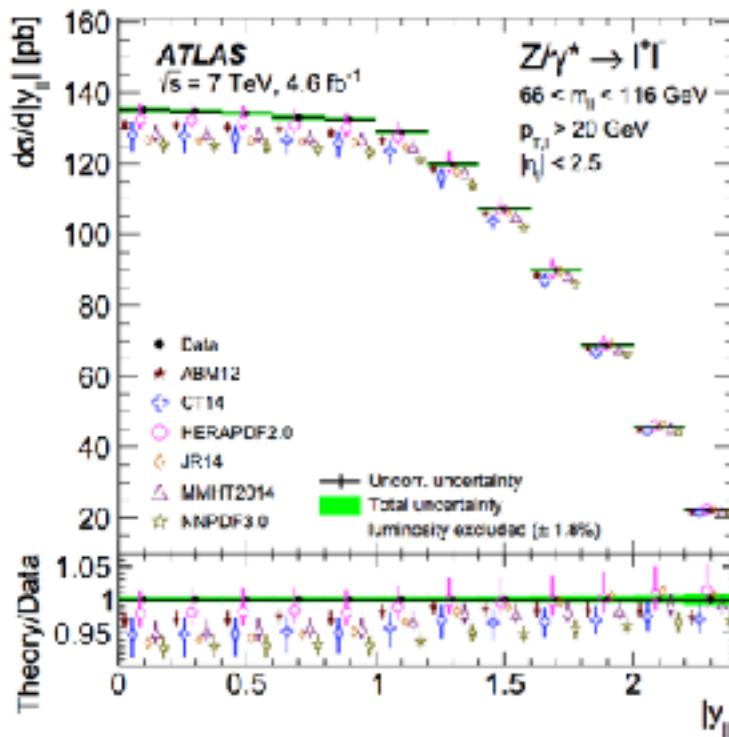


$$\sigma(pp \rightarrow Z) = u\bar{u} + d\bar{d} + s\bar{s}$$

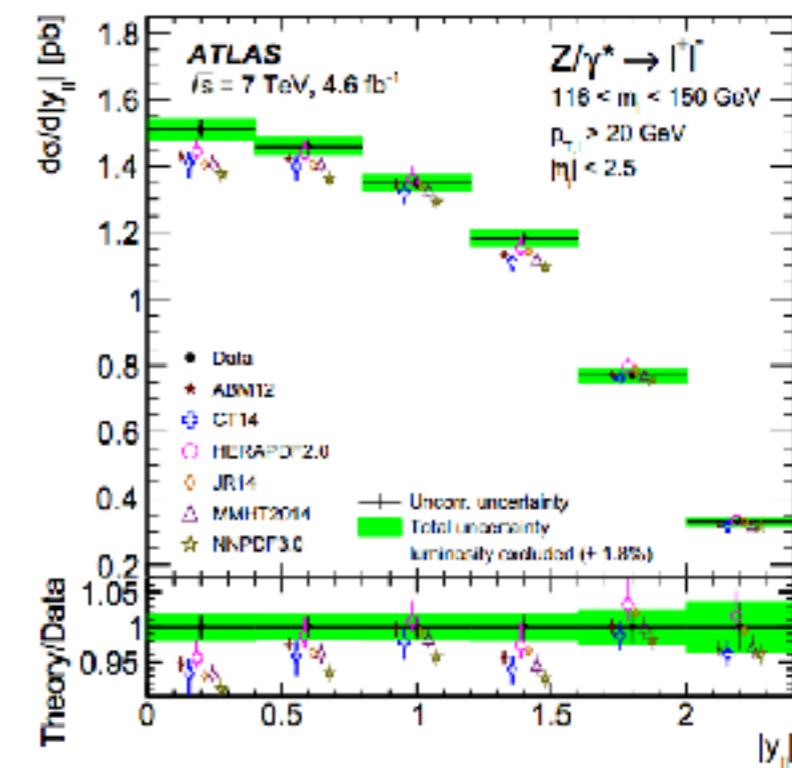
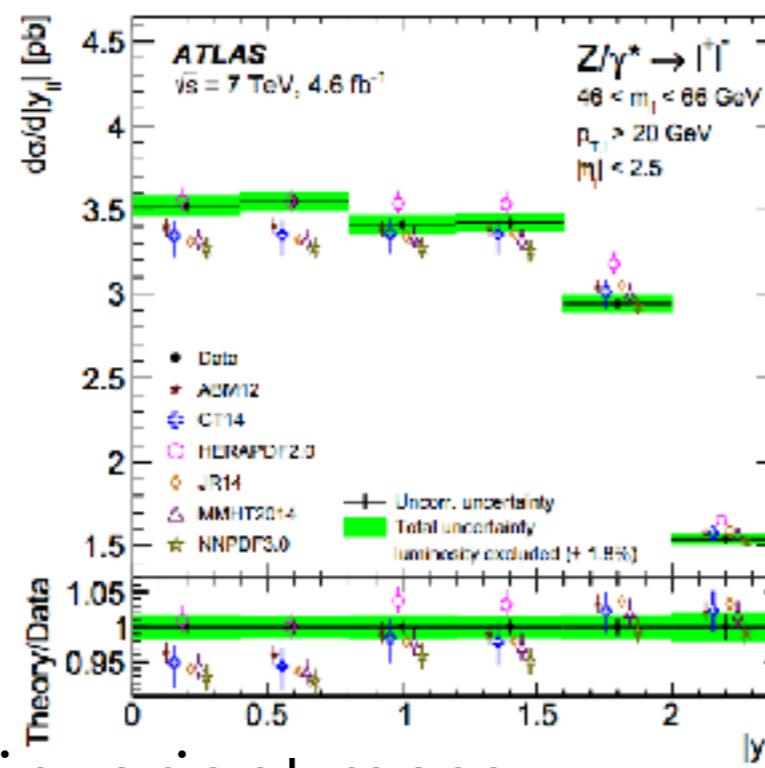
$$\sigma(pp \rightarrow W^+) = u\bar{d} + c\bar{s}$$

$$\sigma(pp \rightarrow W^-) = d\bar{u} + s\bar{c}$$

# Z/W production data

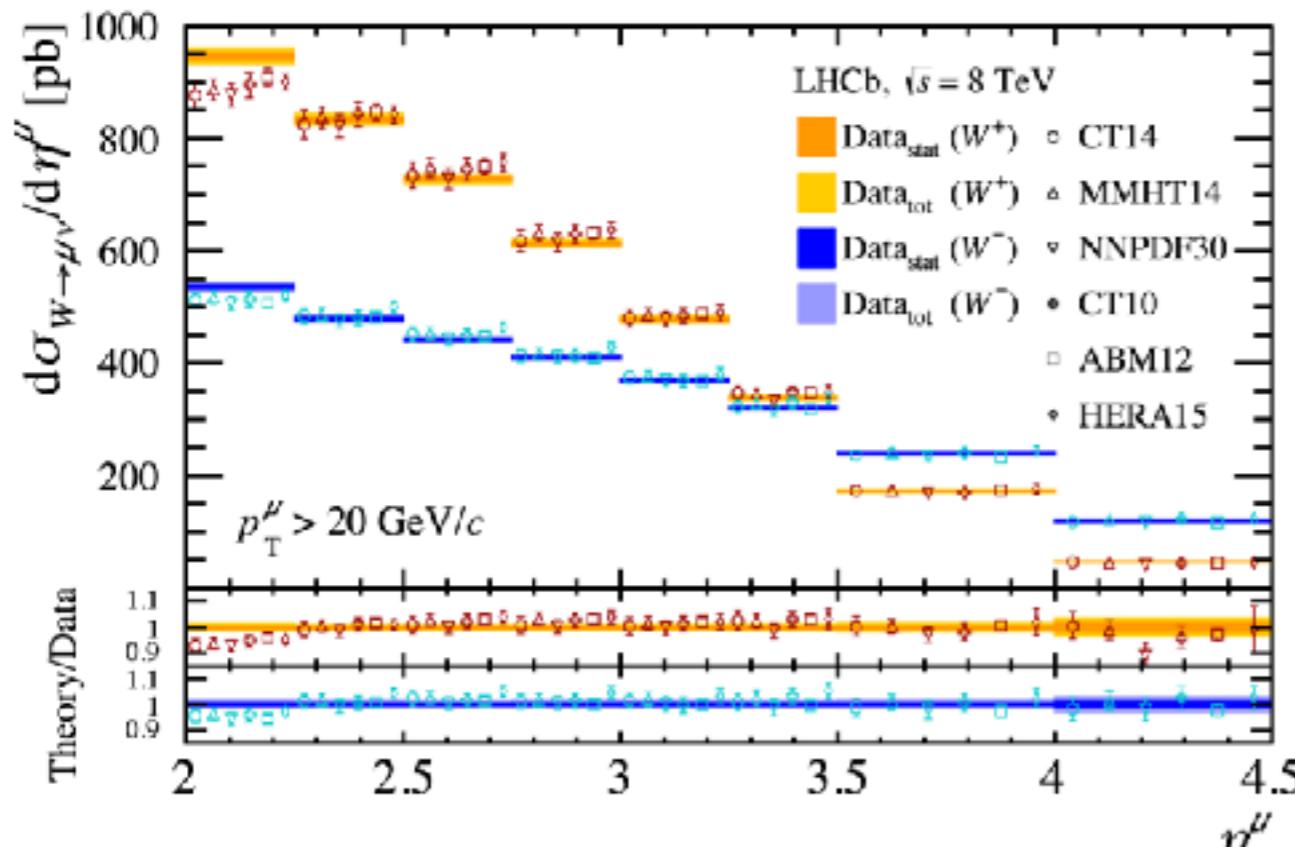


$$x_{1,2} = \frac{M^2}{s} e^{\pm y}$$



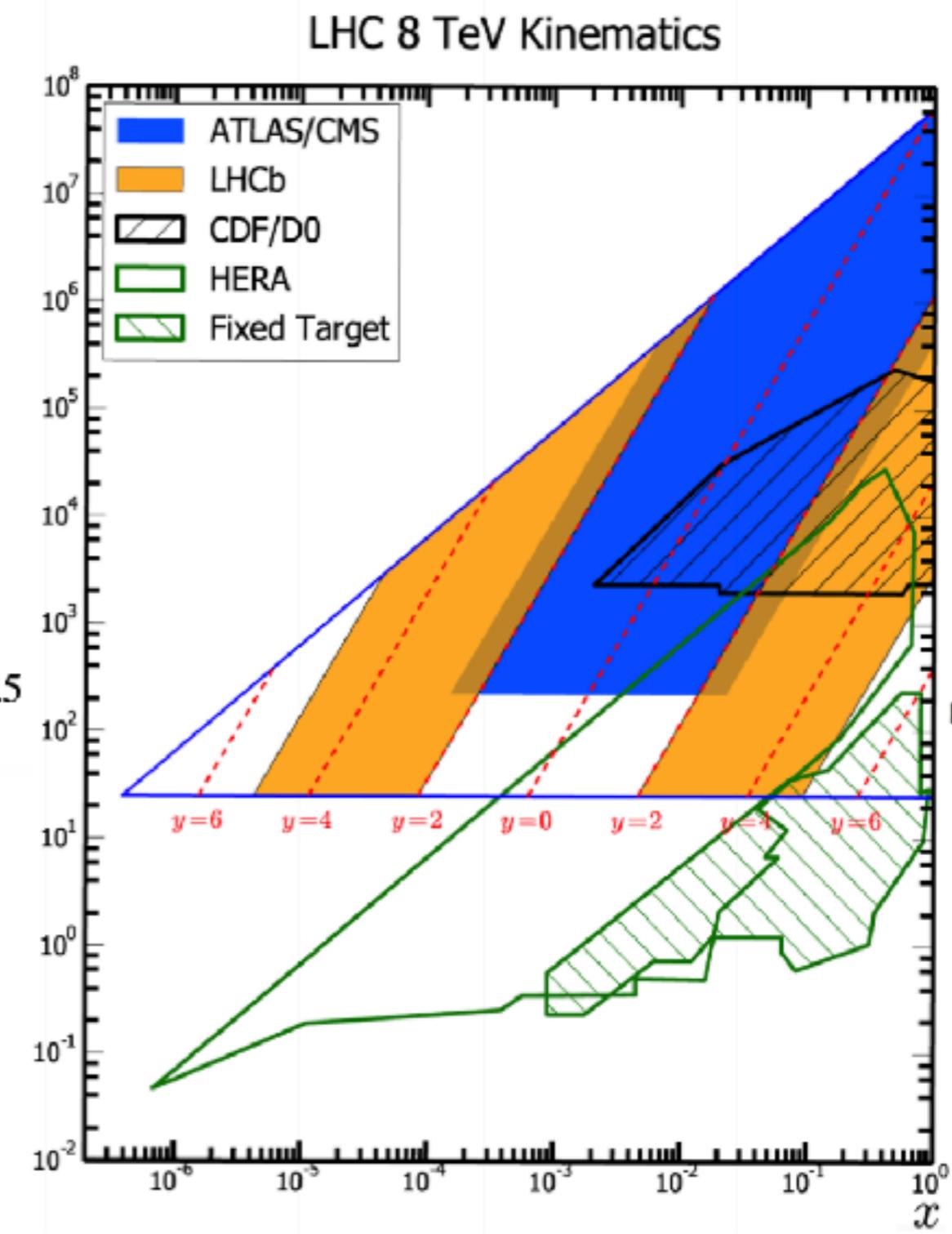
high/low invariant mass

# Z/W production data

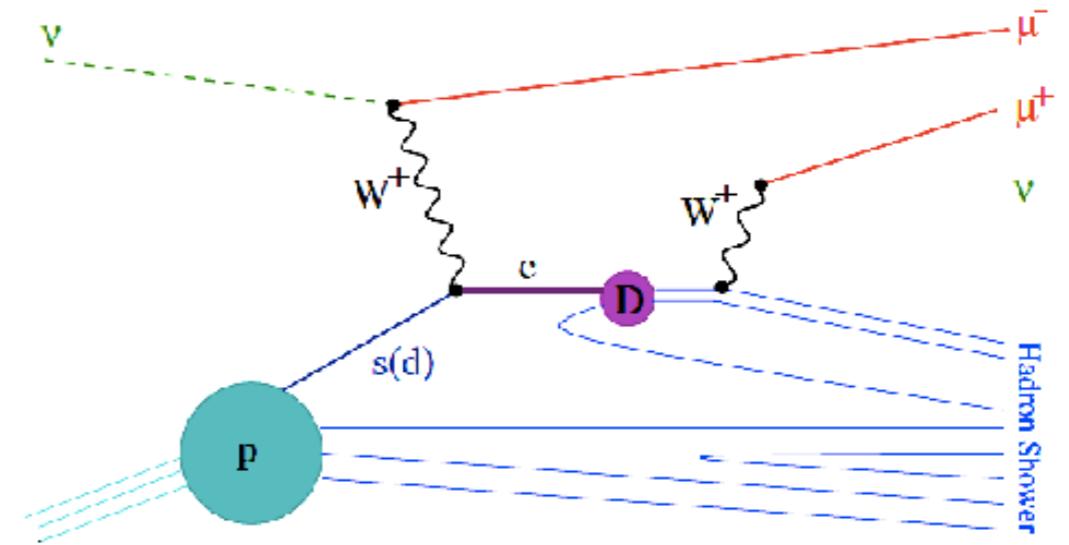
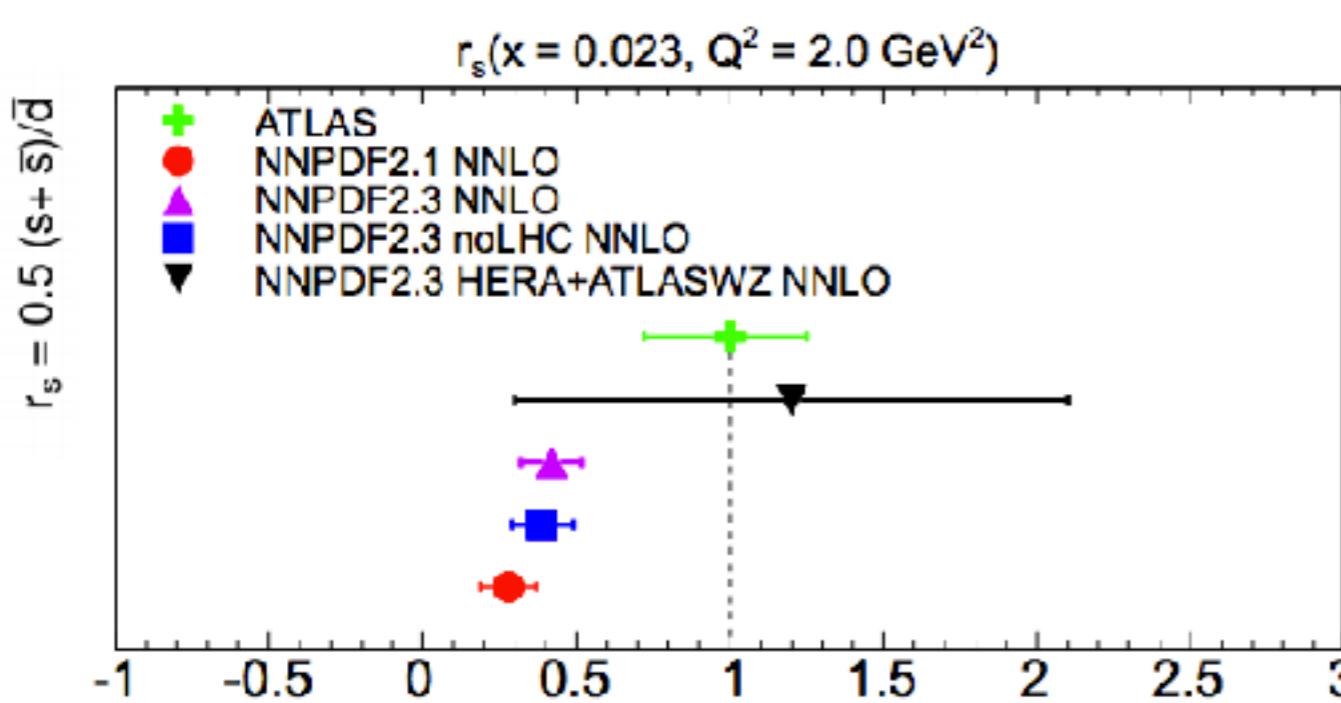
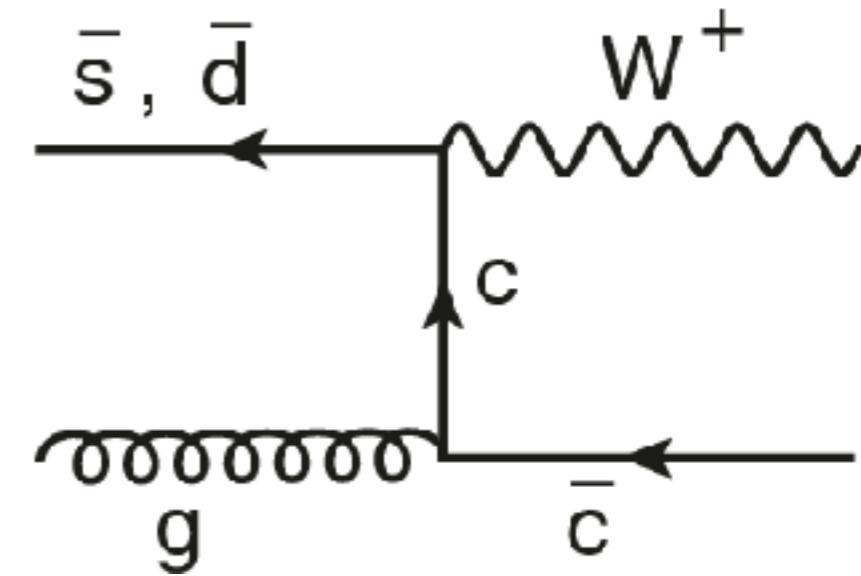
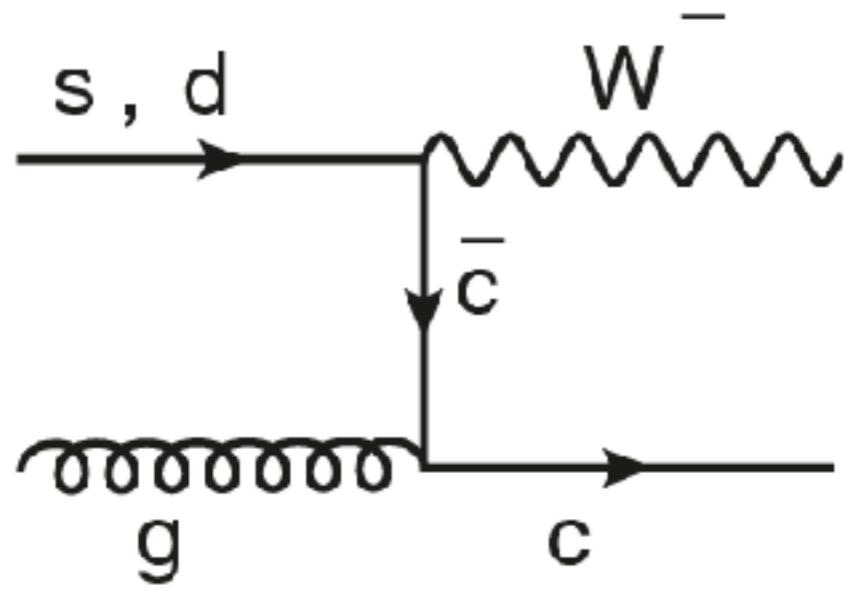


$$x_{1,2} = \frac{M^2}{s} e^{\pm y}$$

LHCb: forward region



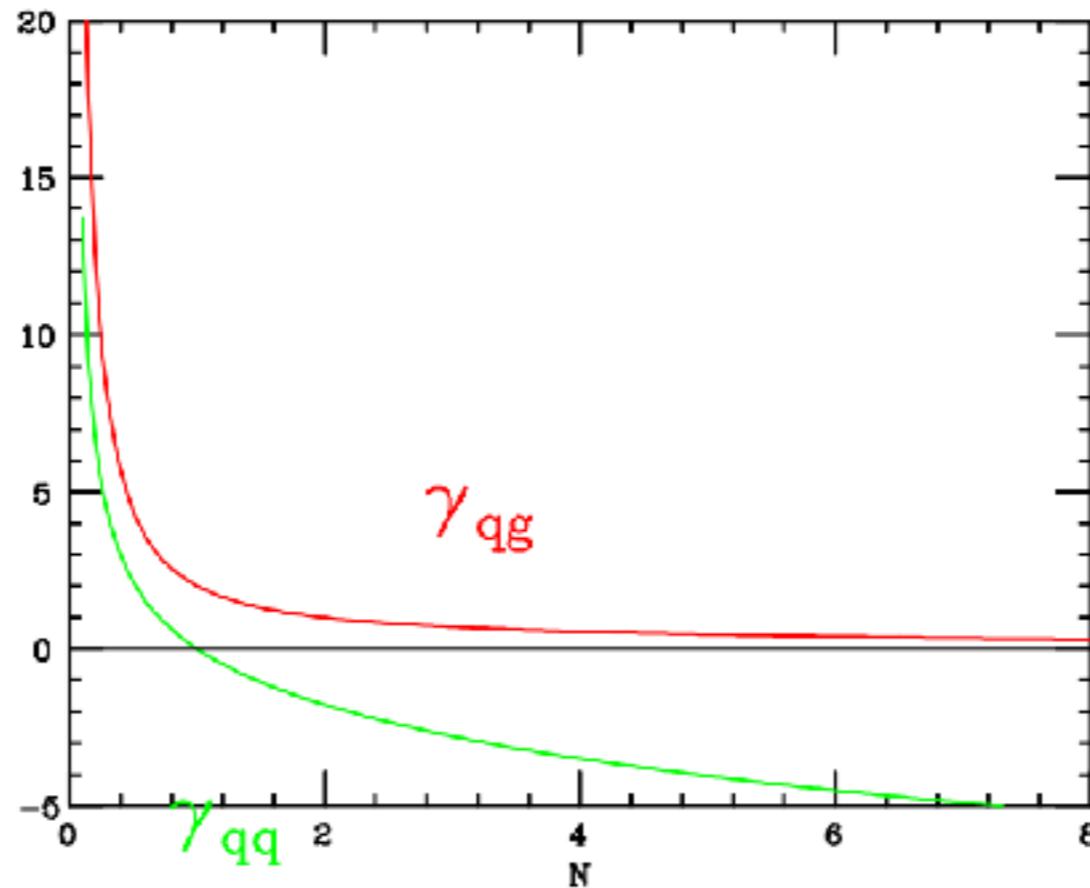
# W+charm data



# Gluon: indirect handle

- Gluon is partially determined by scale dependence of DIS structure functions and Drell-Yan/Vector Boson production

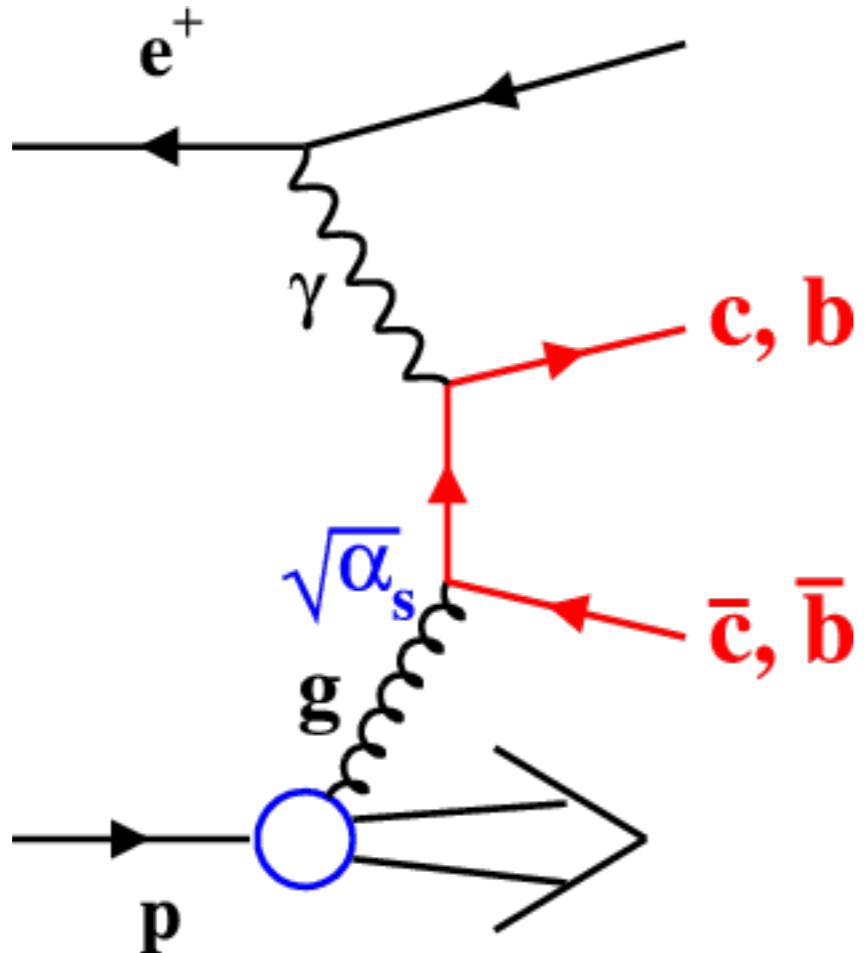
$$\frac{d}{d \log \mu^2} F_2(x, \mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} [P_{qq} \otimes F_2(x, \mu^2) + 2n_f P_{qg} \otimes g(x, \mu^2)]$$



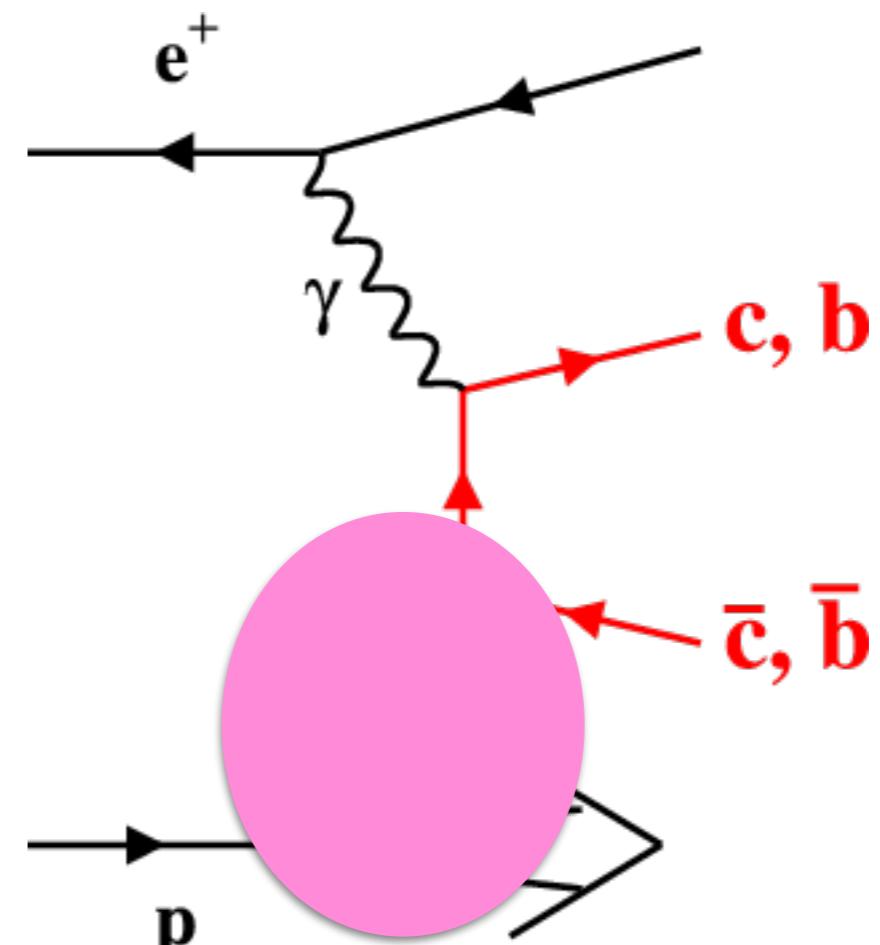
- Mostly determine small-x gluon, large-x gluon hard to determine from DIS+DY only data

# Gluon: indirect handle

- Heavy quarks are produced at threshold inside proton
- Heavy quark production process (at ep and pp colliders) probe gluon
- Dependence on heavy flavour scheme adopted in PDF fitting



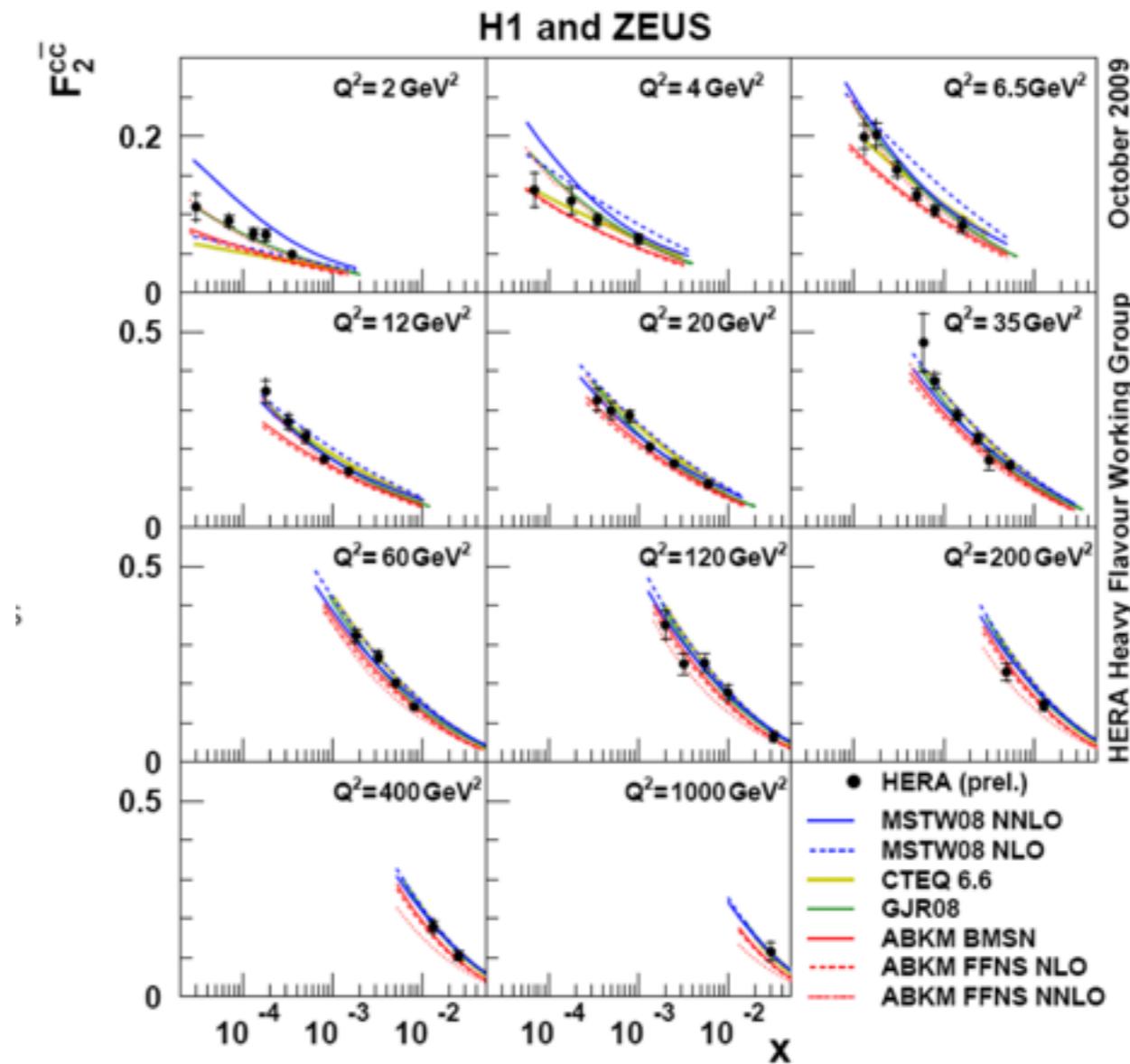
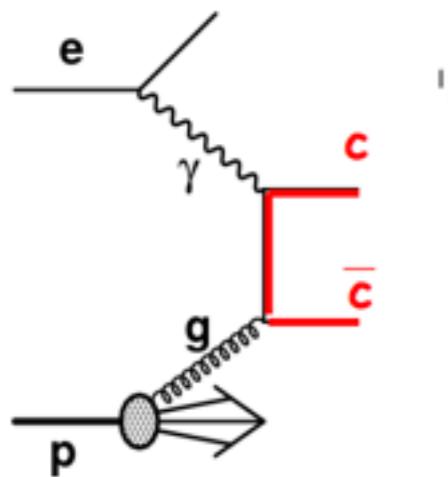
Nf = 3,4



Nf = 5

# Gluon: indirect handle

- Heavy quarks are produced at threshold inside proton
- Heavy quark production process (at ep and pp colliders) probe gluon
- Dependence on heavy flavour scheme adopted in PDF fitting



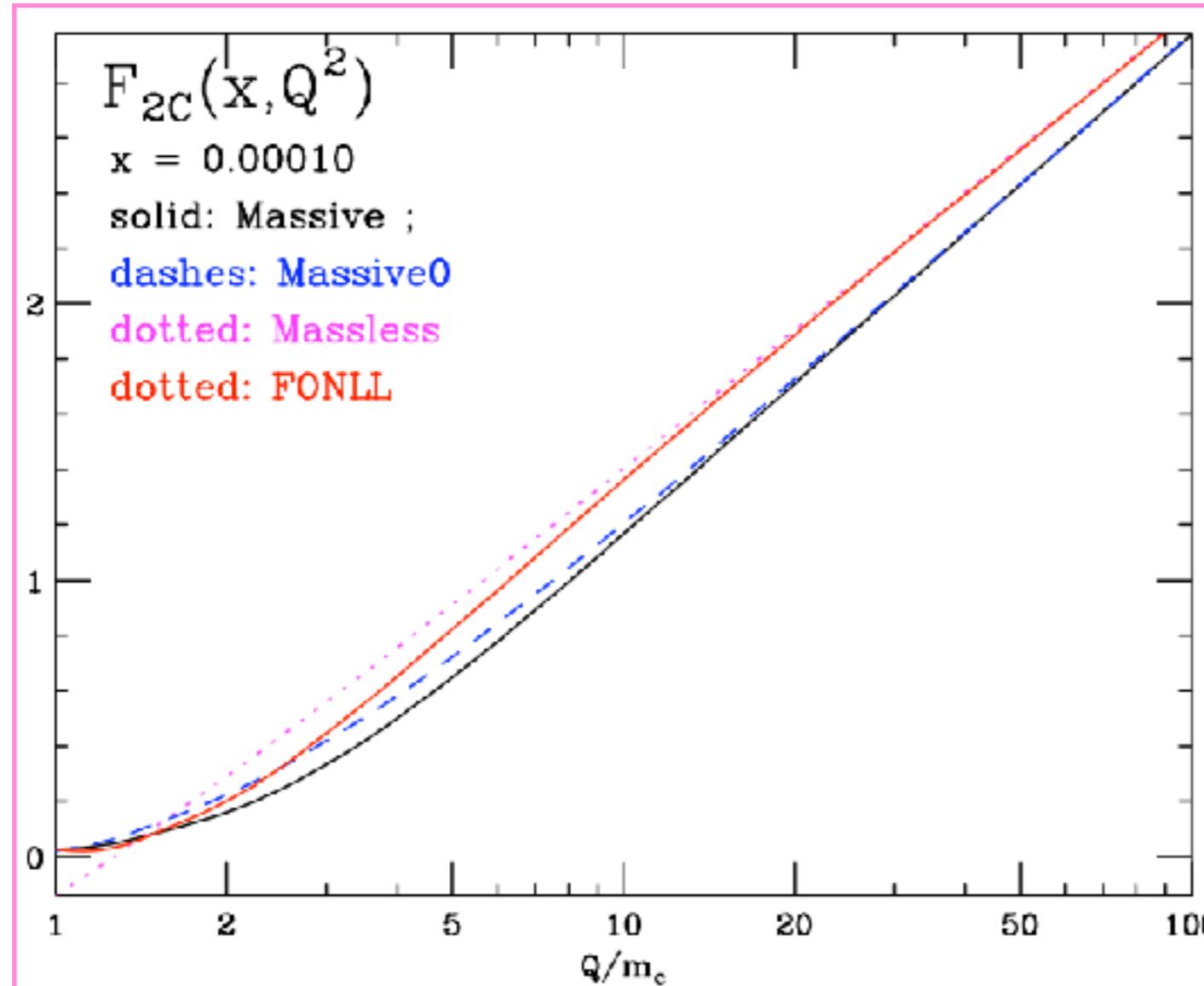
# Intermission: heavy flavour schemes

- Charm, Bottom and Top have mass  $\gg \Lambda_{\text{QCD}}$  - heavy quarks (HQ)
- The presence of a new scale,  $m_Q$ , makes pert QCD calculations more challenging
- Two well understood schemes:
  - Assume heavy quark effectively massless for  $Q > m_Q$   
HQ becomes active massless parton above threshold
  - Heavy quarks retain their mass for all  $Q$   
HQ is not a parton, it is a final state particle
- However in PDF fits we have all scales. General-Mass Variable-Flavor-Number schemes allow to match between the zero-mass and the massive scheme
- Many schemes available

e.g. FONLL

$$\begin{aligned}\sigma^{(\text{FONLL})} &= \sigma^{(4)} + \sigma^{(5)} - \text{double counting} \\ &= \mathcal{L}_{ij}(x_1, x_2, \mu^2) \otimes \sum_p^N \left( \alpha_s^{(5)}(\mu^2) \right)^p \\ &\times \left\{ \mathcal{B}_{ij}^{(p)} \left( x_1, x_2, \frac{\mu^2}{m_b^2} \right) + \sum_{k=0}^{\infty} \mathcal{A}_{ij}^{(p),(\text{k})}(x_1, x_2) \left( \alpha_s^{(5)}(\mu^2) L \right)^k \right\} \\ &- \text{double counting}\end{aligned}$$

# Intermission: heavy flavour schemes



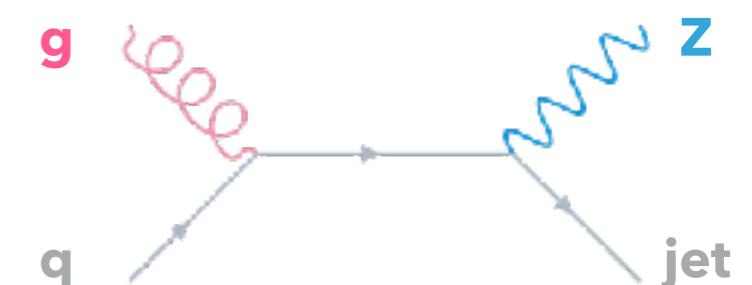
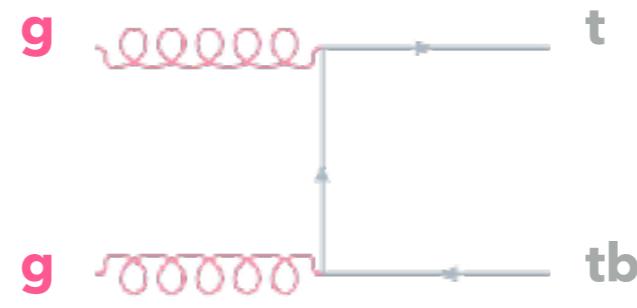
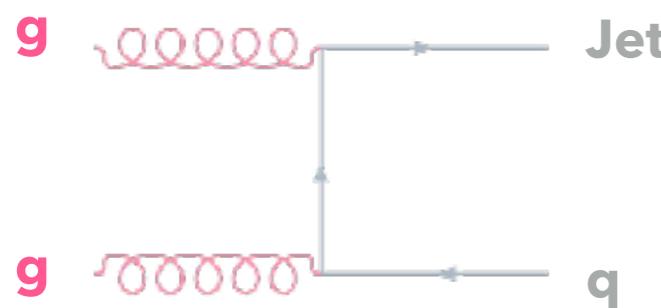
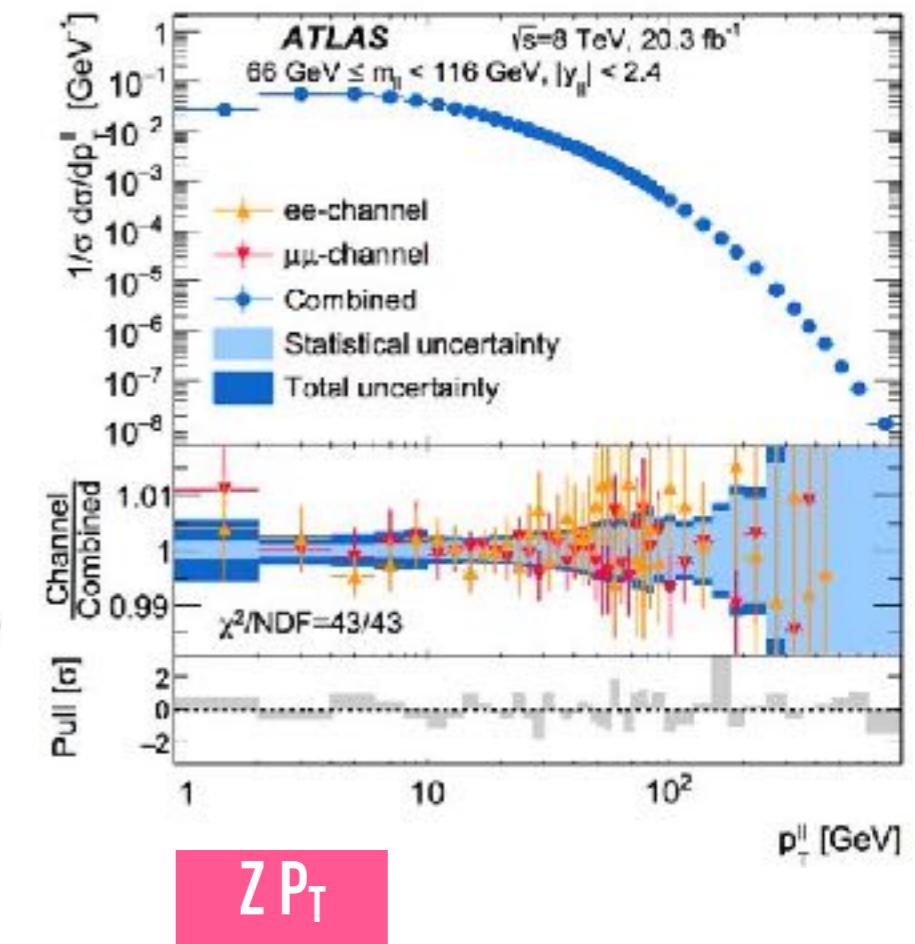
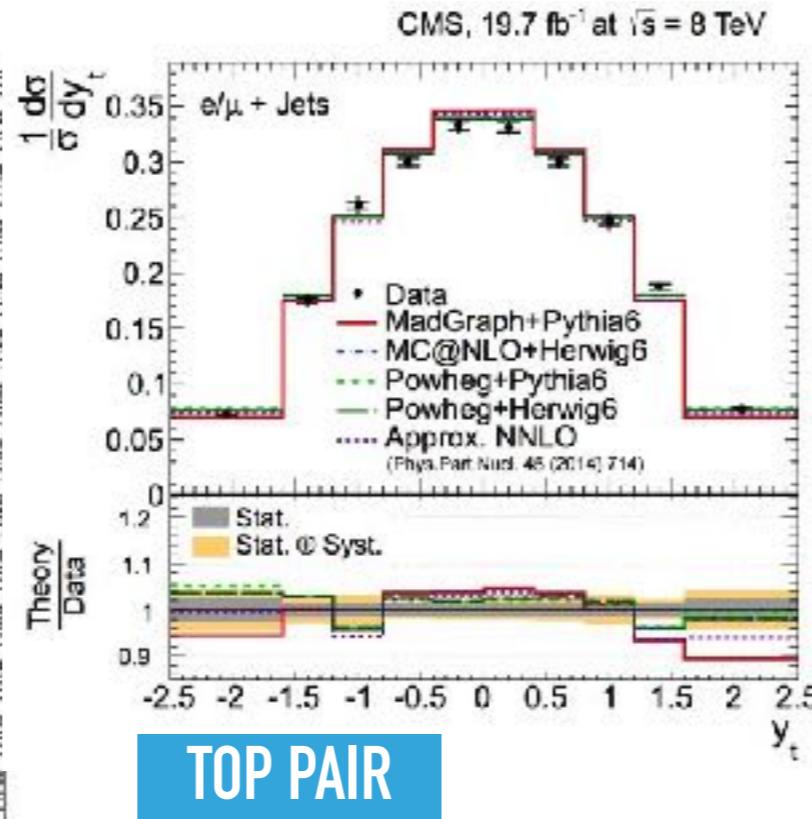
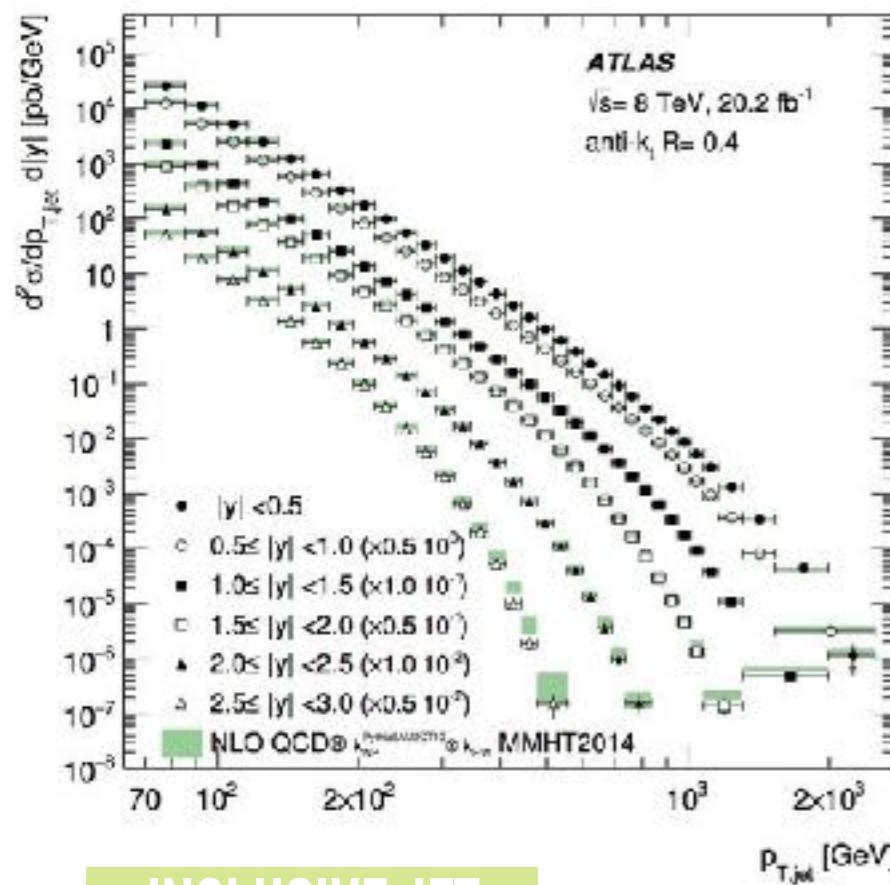
- heavy quarks (HQ)  
D calculations more challenging  
massless for  $Q > m_Q$   
above threshold  
 $Q^2$   
article  
I-Mass Variable-Flavor-Number  
and the massive scheme

- Many schemes available

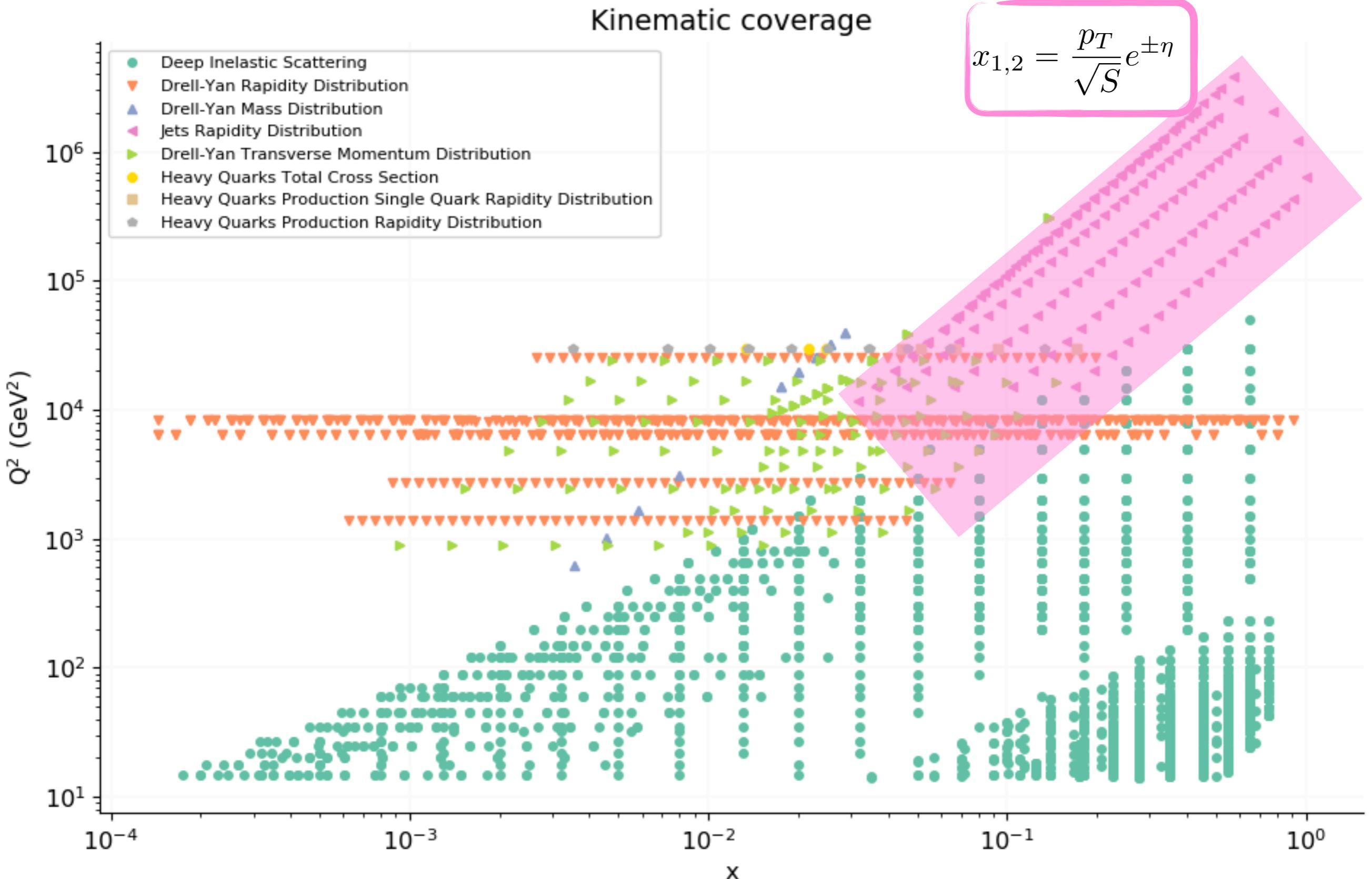
e.g. FONLL

$$\begin{aligned}\sigma^{(FONLL)} &= \sigma^{(4)} + \sigma^{(5)} - \text{double counting} \\ &= \mathcal{L}_{ij}(x_1, x_2, \mu^2) \otimes \sum_p^N \left( \alpha_s^{(5)}(\mu^2) \right)^p \\ &\times \left\{ \mathcal{B}_{ij}^{(p)} \left( x_1, x_2, \frac{\mu^2}{m_b^2} \right) + \sum_{k=0}^{\infty} \mathcal{A}_{ij}^{(p),(\text{k})}(x_1, x_2) \left( \alpha_s^{(5)}(\mu^2) L \right)^k \right\} \\ &- \text{double counting}\end{aligned}$$

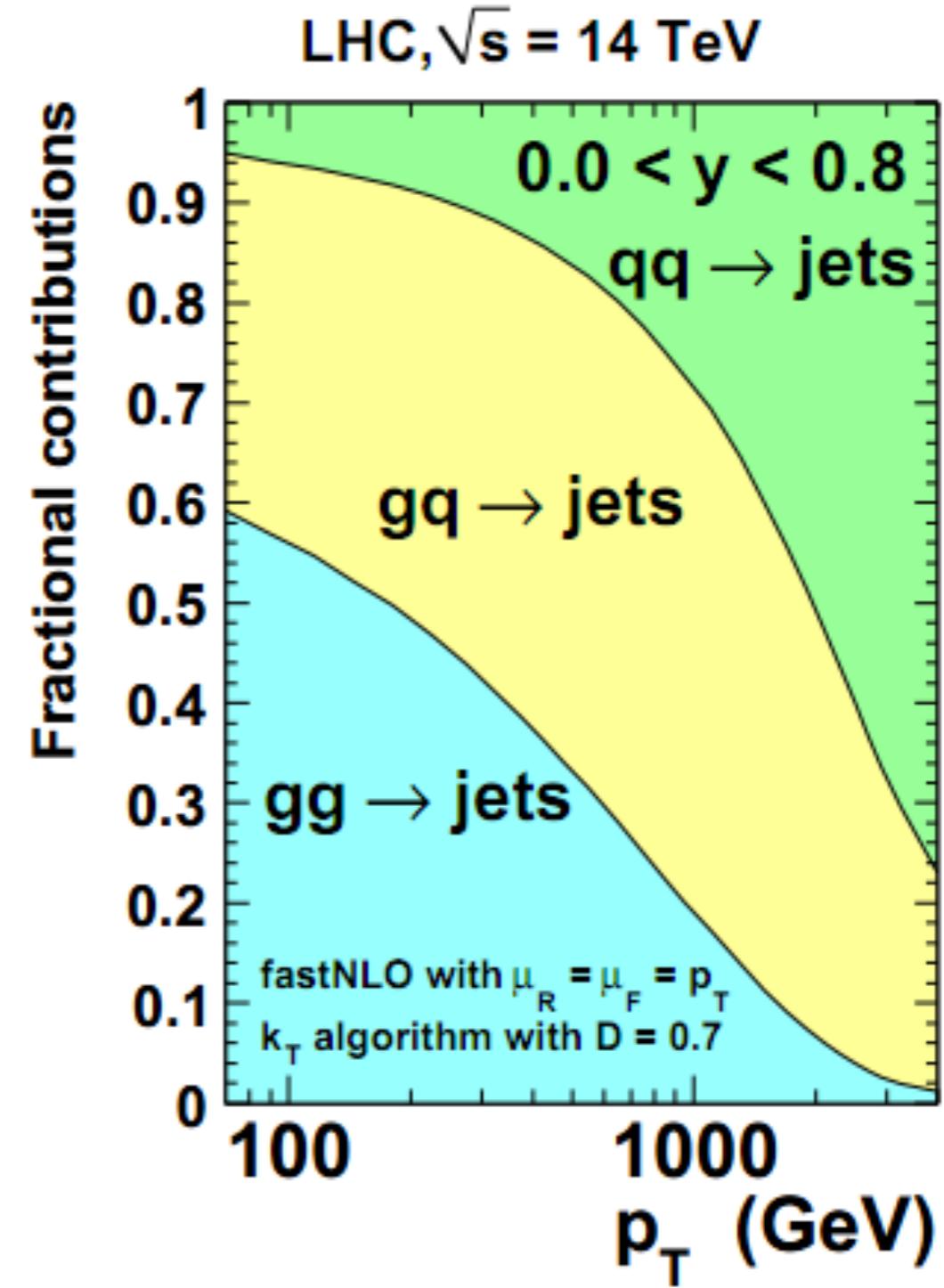
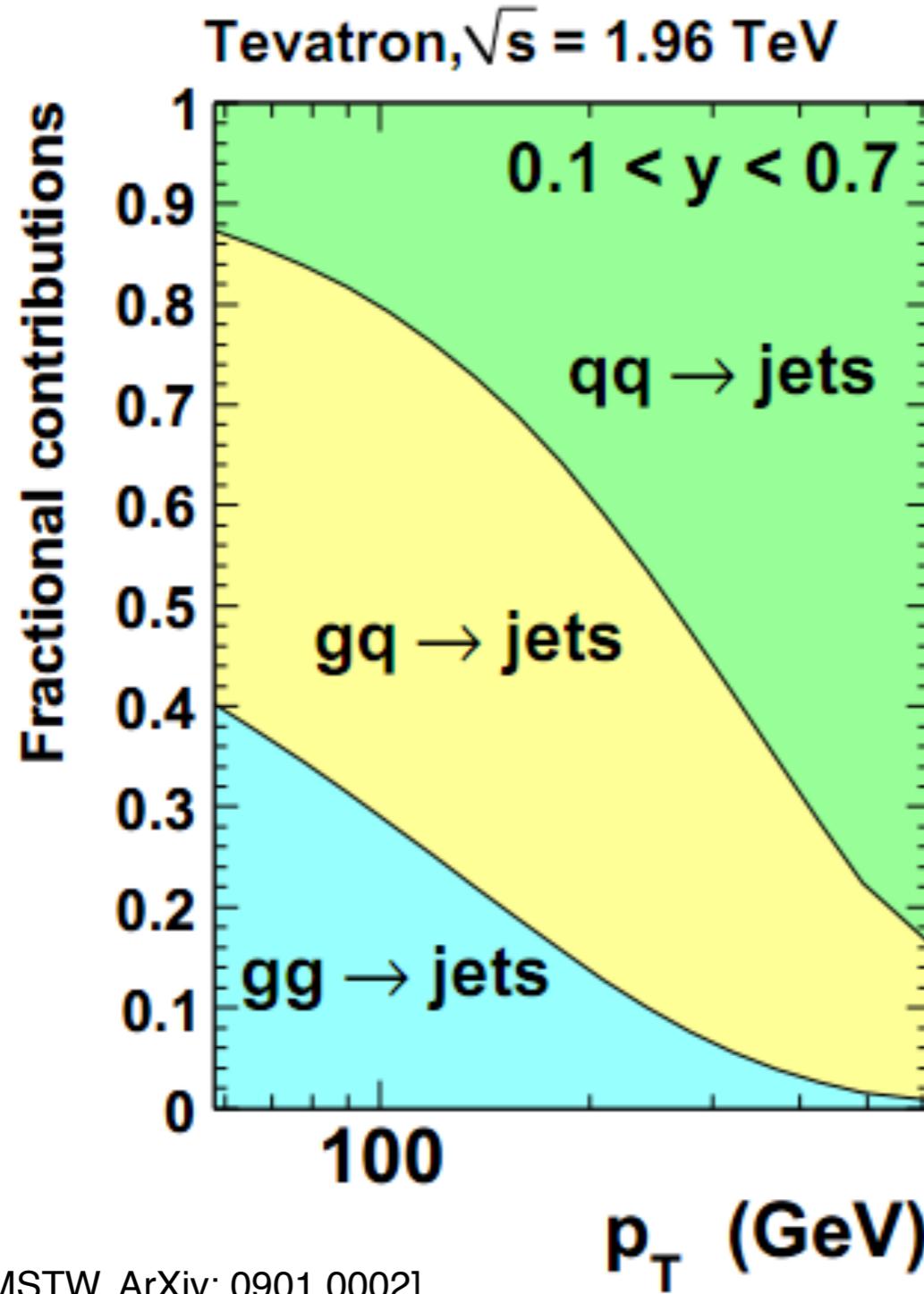
# Gluon: direct handle



# Gluon: jets data



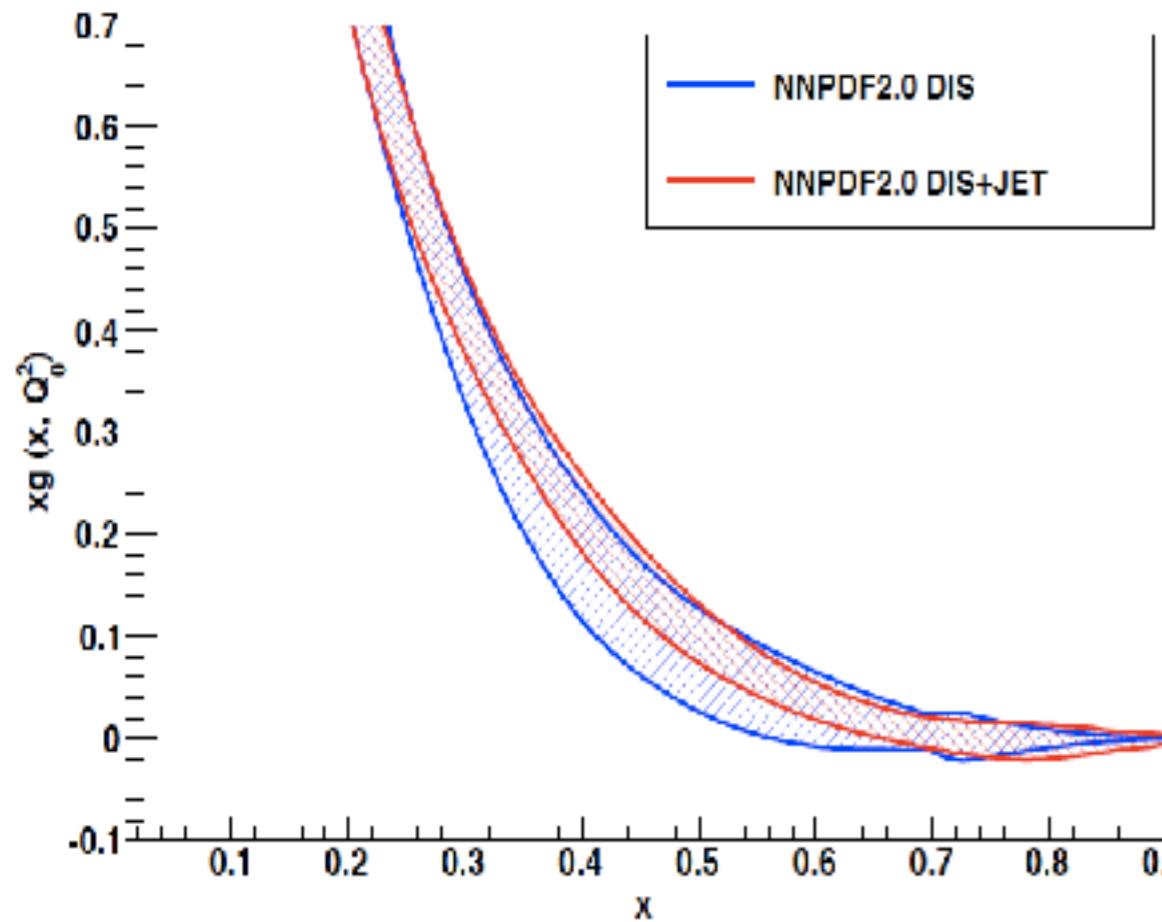
# Gluon: jets data



# Gluon: jets data

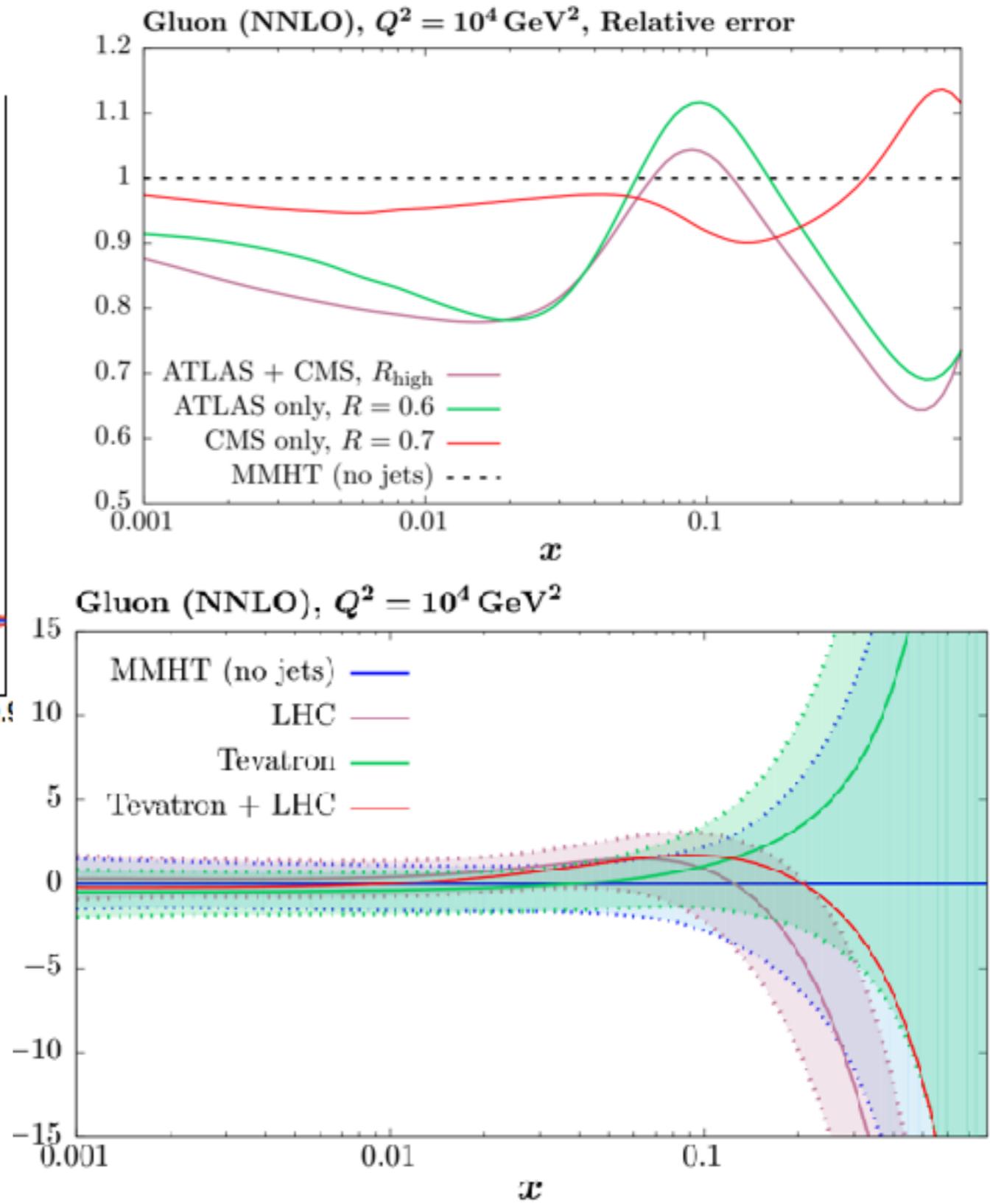
LHC jet data

Tevatron jet data



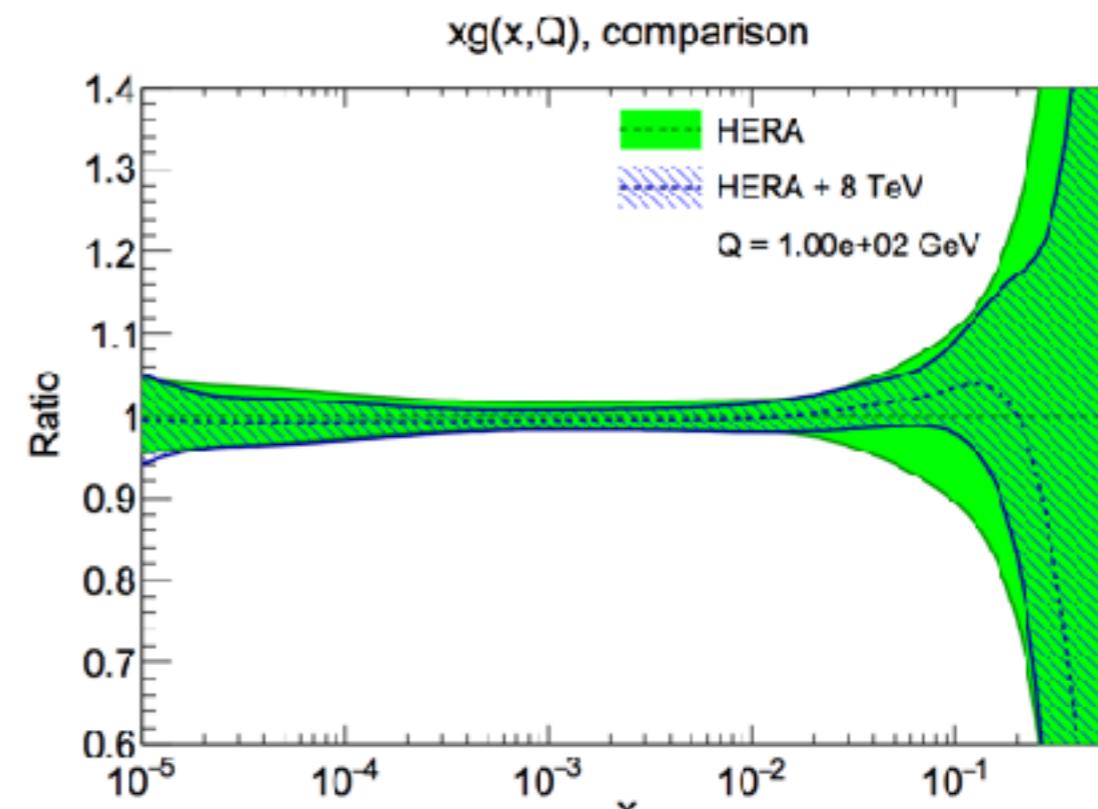
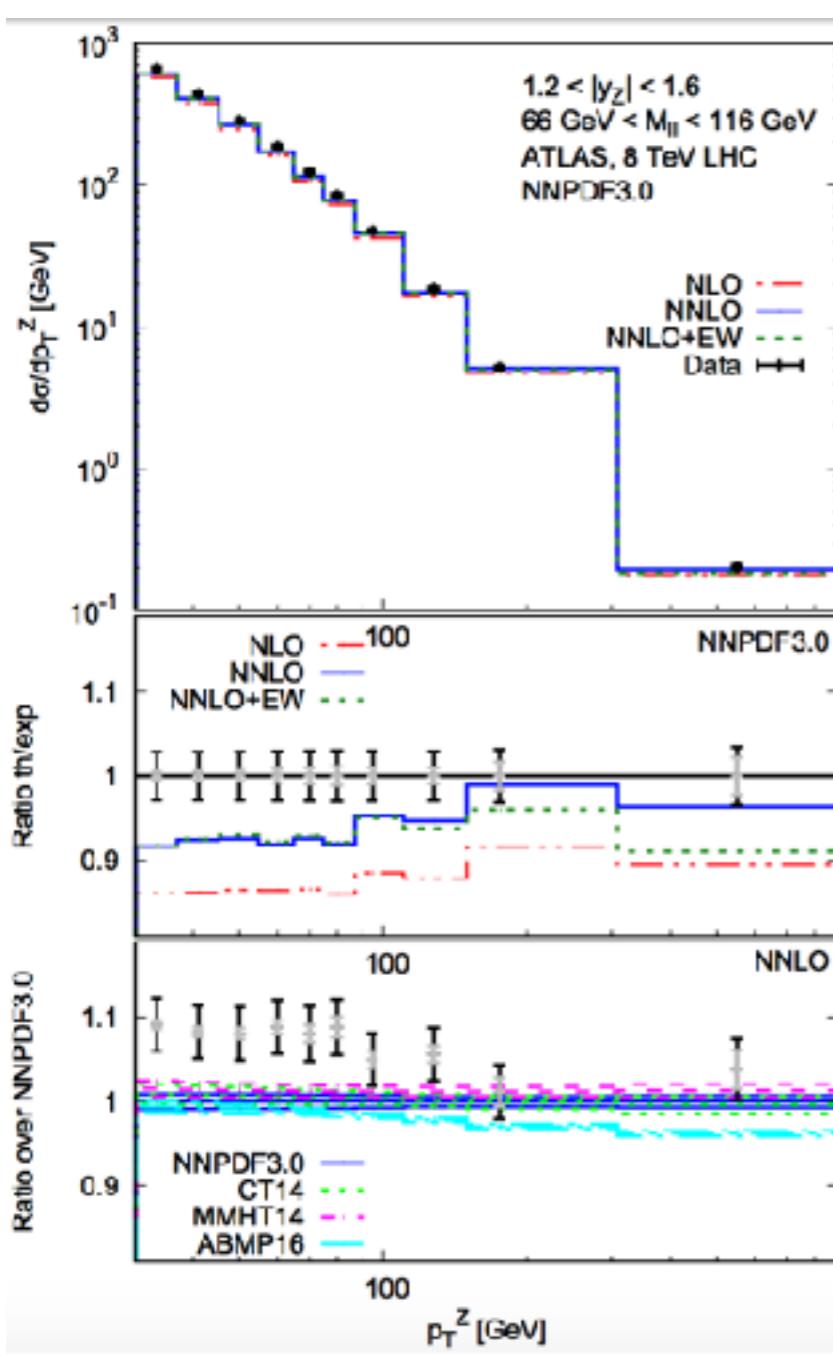
[Ball et al, ArXiv: 1002.4407]

[Harland-Lang et al, ArXiv: 1711.05757]



# Gluon: Z transverse momentum

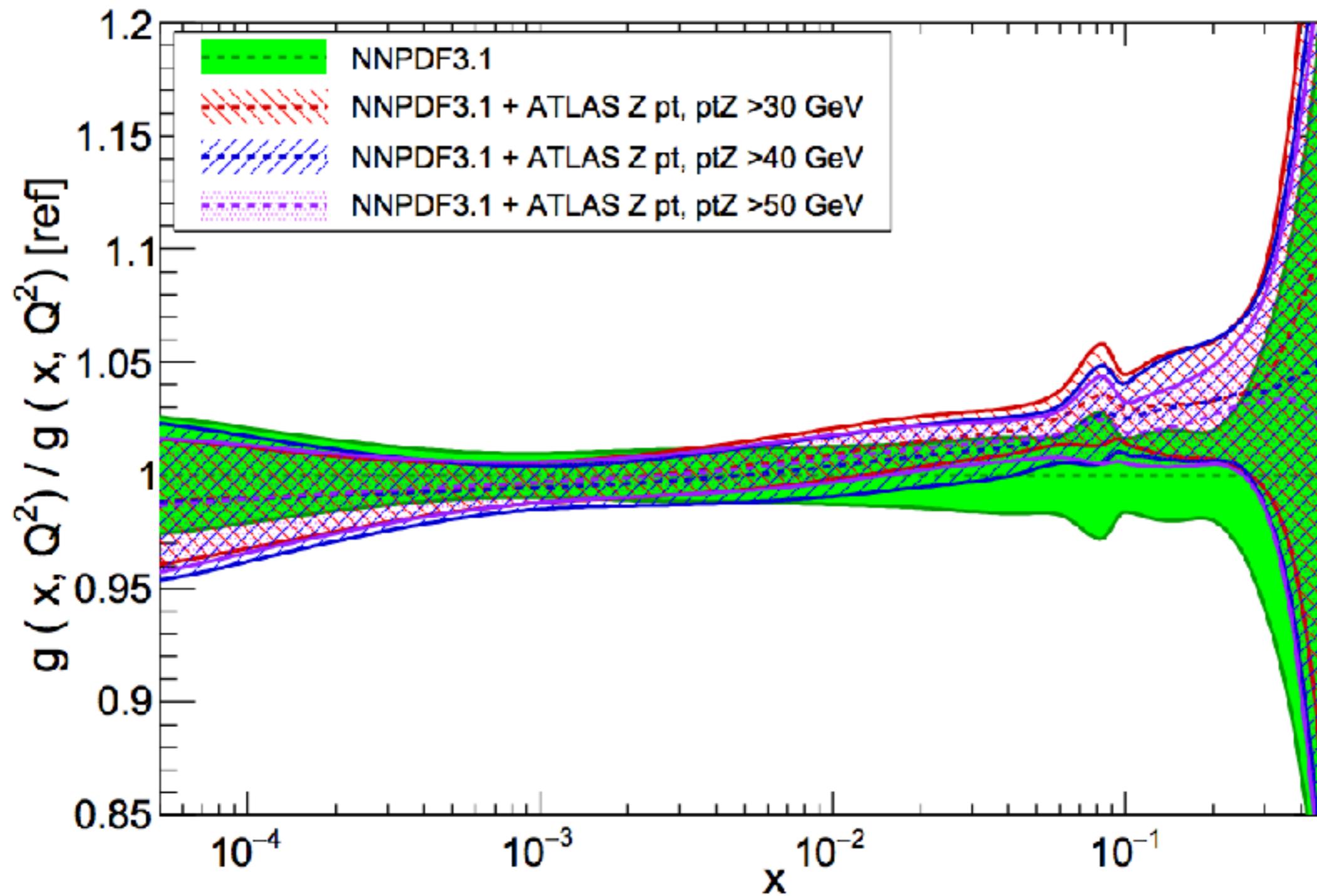
- Experimental precision < 1% up to  $p_T \sim 200$  GeV
- Data hugely dominate by correlated systematic uncertainties
- Interesting case-study to probe current theory-experiment frontier



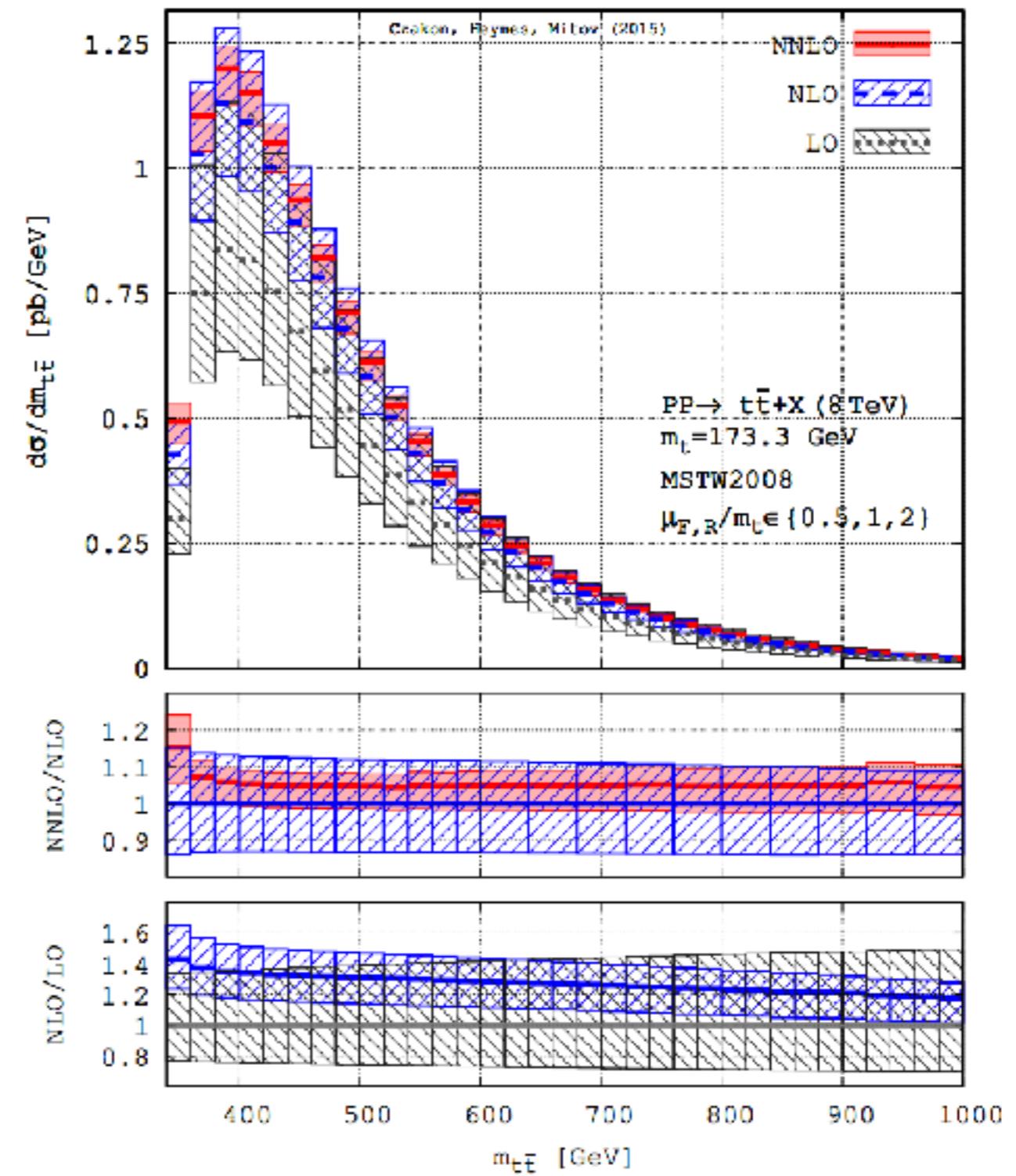
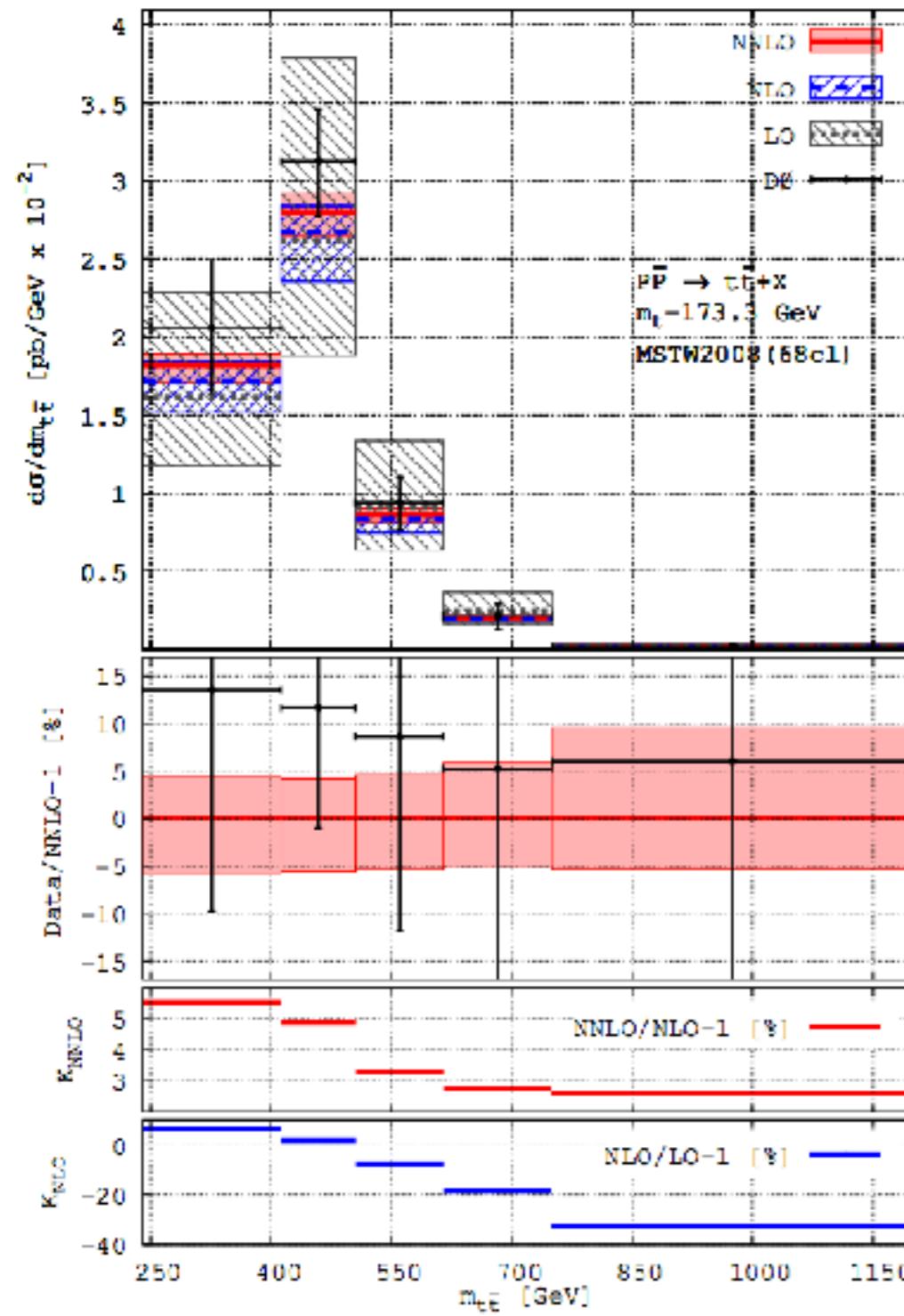
- ▶ Data/Theory comparison not so intuitive for correlation-dominated data
- ▶ Fluctuation in NNLO predictions (0.5 - 1%) had to be accounted for as extra nuisance parameter to get a good fit of such precise data

# Gluon: Z transverse momentum

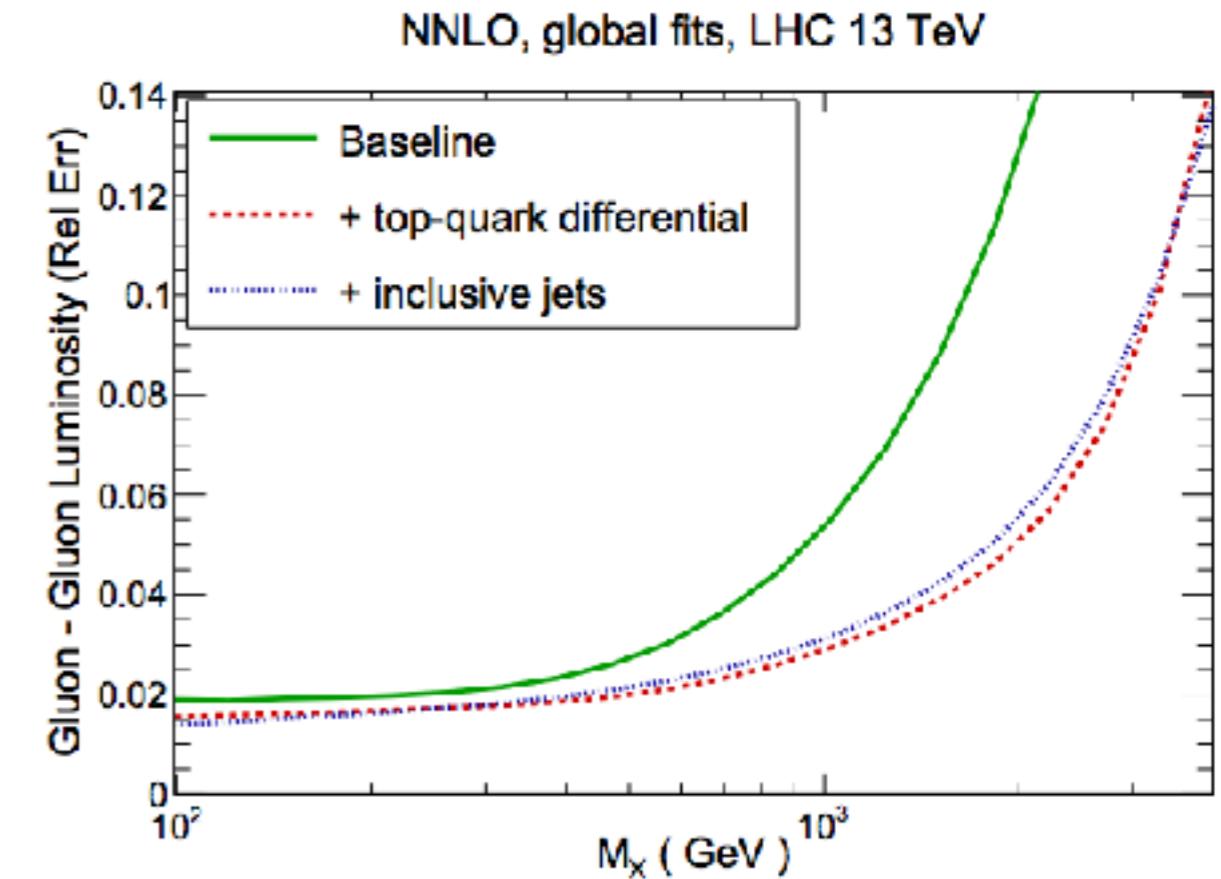
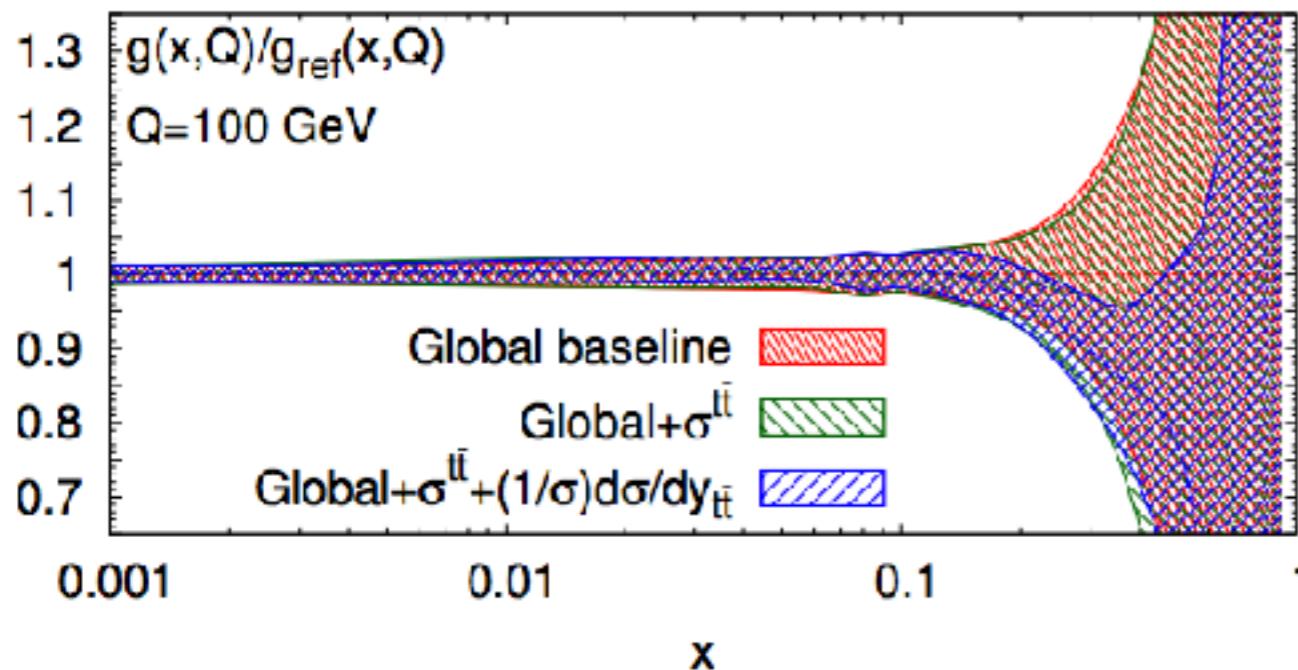
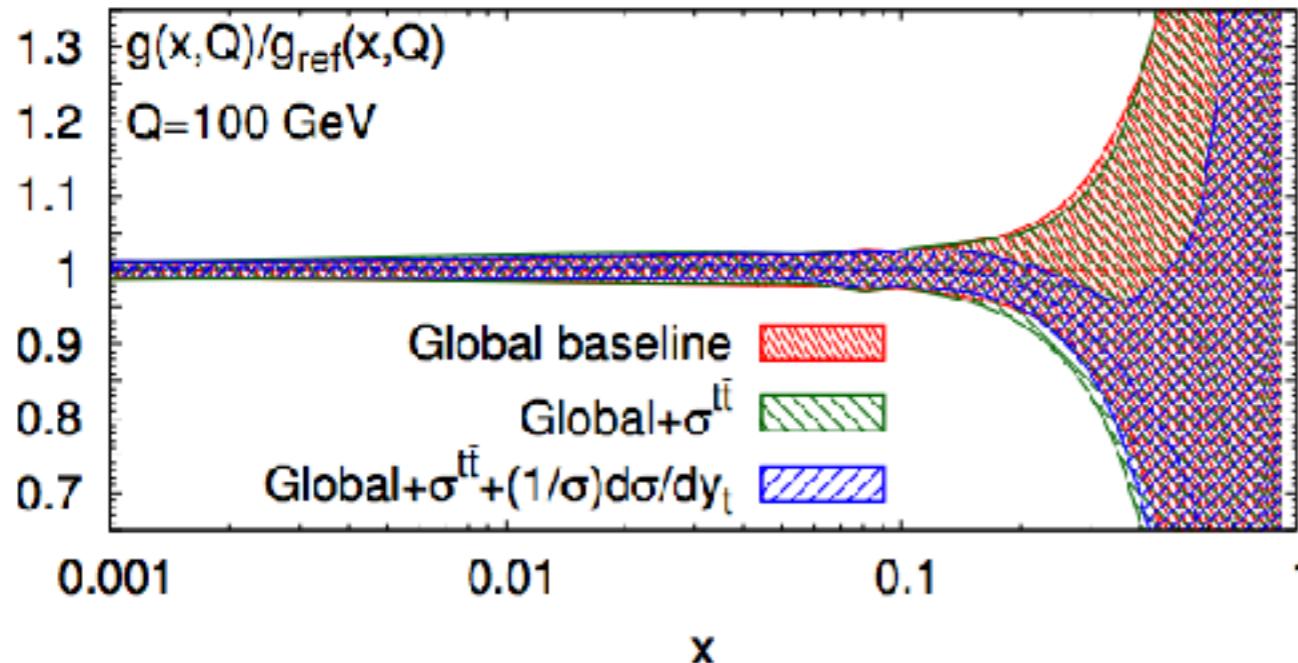
NNLO,  $Q^2=10^4 \text{ GeV}^2$



# Gluon: top pair production



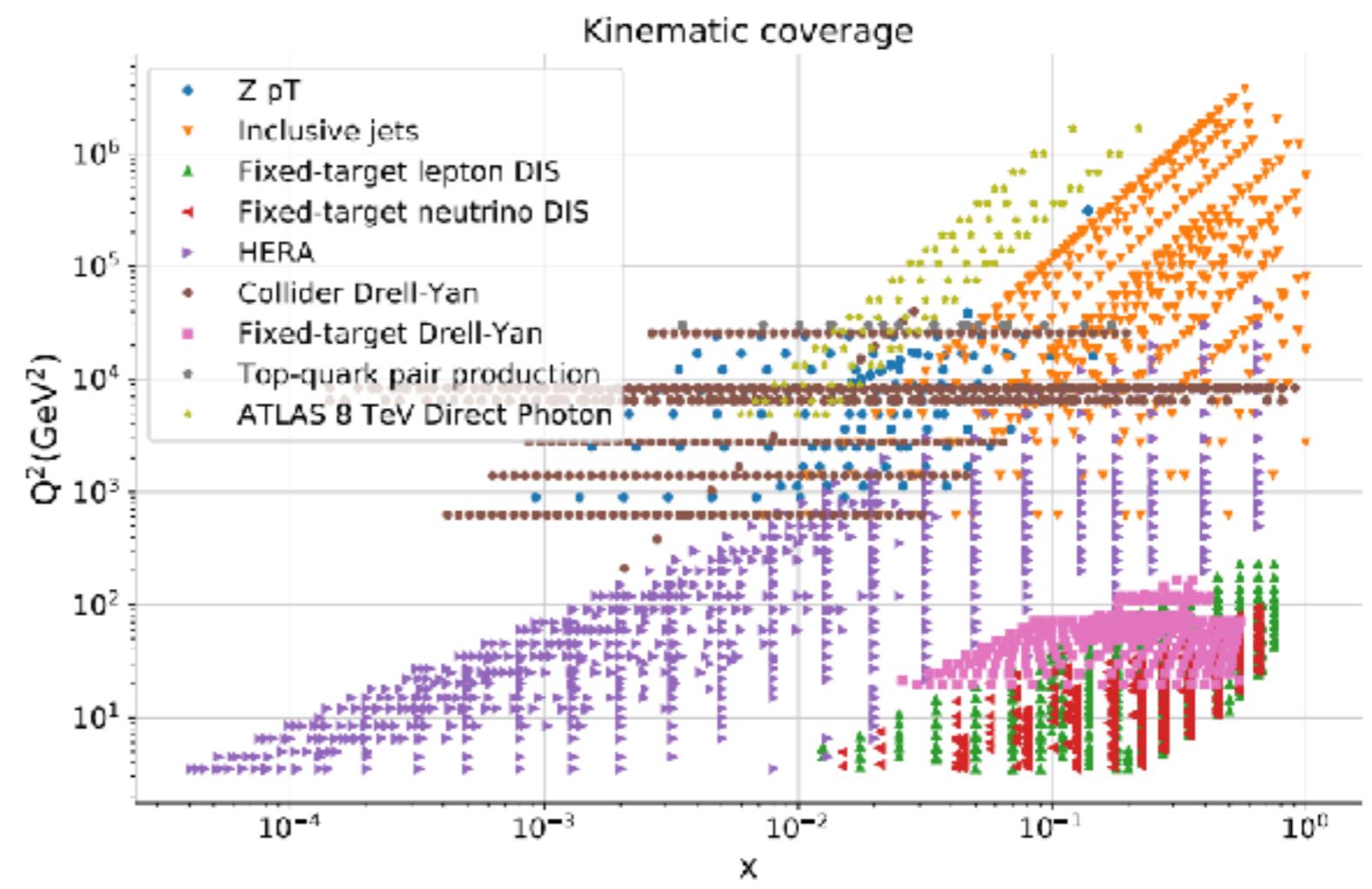
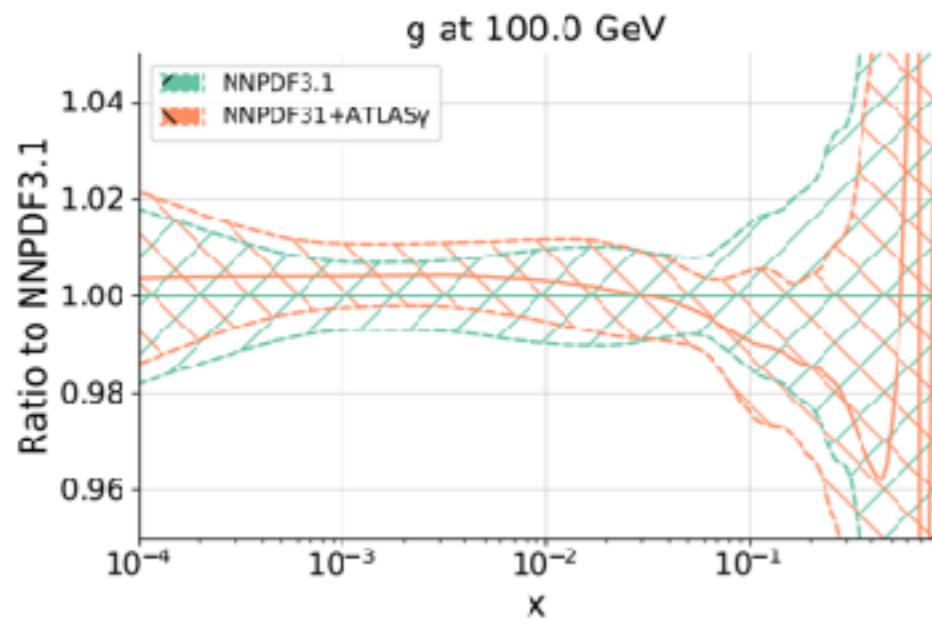
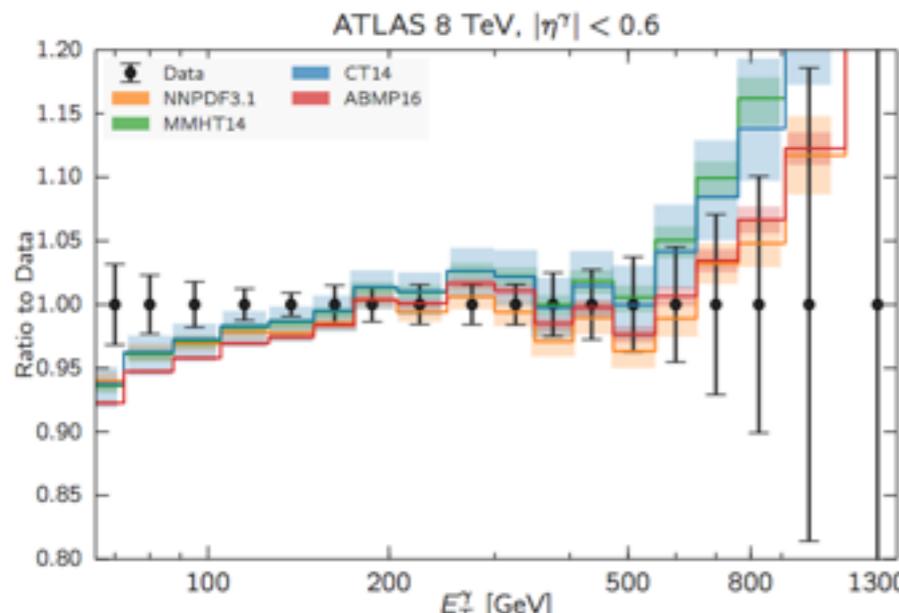
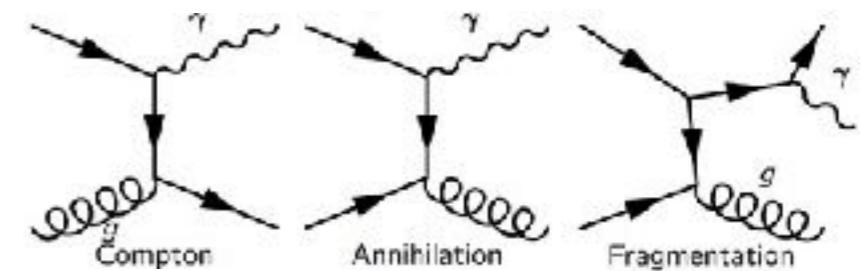
# Gluon: top pair production



- Most constraining is inclusion of  $y_t$  list from ATLAS and  $y_{tt}$  from CMS jointly with total xsec
- Competitive reduction of gluon uncertainty with jets measurement
- Slight tension between ATLAS and CMS in NNPDF3.1 ( $\chi^2_{\text{ATLAS}} \sim 1.6$ ,  $\chi^2_{\text{CMS}} \sim 0.9$ )

# Gluon: direct photon production

- Prompt photon production directly sensitive to the gluon-quark luminosity via Compton scattering
- Isolated prompt photon data known at NNLO [Campbell et al 1612.04333] and accurately measured by ATLAS



Campbell et al 1802.03021

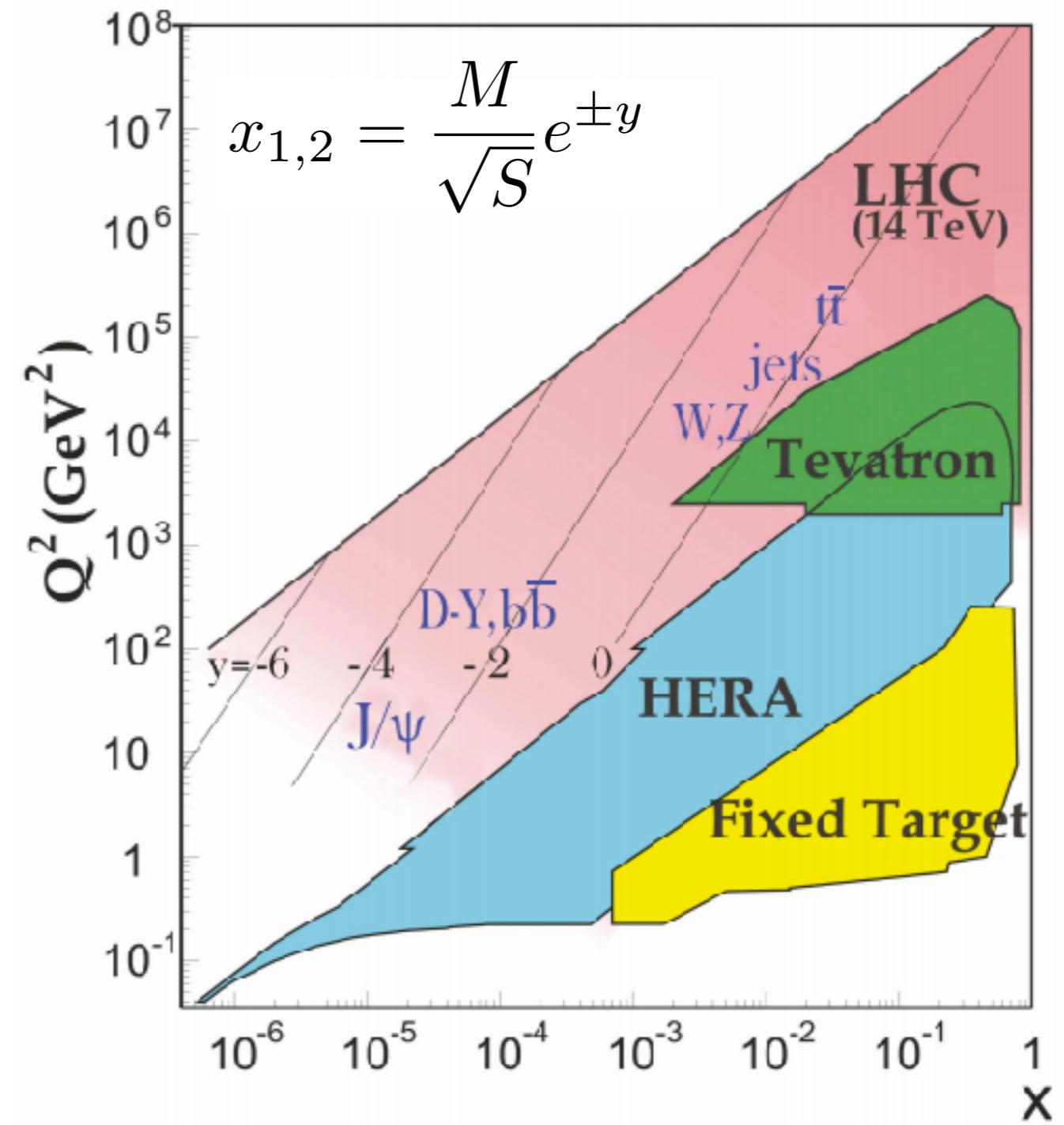
# To summarise

GLUON

- { Inclusive jets and dijets  
**(medium/large x)**
- Isolated photon and  $\gamma$ +jets  
**(medium/large x)**
- Top pair production **(large x)**
- High  $p_T V(+\text{jets})$  distribution  
**(medium x)**

QUARKS

- { High  $p_T V(+\text{jets})$  ratios  
**(medium x)**
- W and Z production  
**(medium x)**
- Low and high mass Drell-Yan  
**(small and large x)**
- $W_c$  **(strangeness at medium x)**



# Parton Luminosities

- A quick and easy way to assess the mass and the collider dependence of production cross sections at hadron-hadron colliders is to use Parton Luminosities
- At leading order in QCD (parton model)

$$\hat{\sigma}_{ab \rightarrow X} = C_X \delta(x_a x_b S - M^2)$$

$$\sigma_{pp \rightarrow X} = \int_0^1 dx_a dx_b f_a(x_a, M^2) f_b(x_b, M^2) \hat{\sigma}_{ab \rightarrow X}$$

- Thus

$$\begin{aligned}\sigma_{pp \rightarrow X} &= C_X \int_0^1 dx_a dx_b f_a(x_a, M^2) f_b(x_b, M^2) \delta(x_a x_b S - M^2) \\ &= \frac{C_X}{S} \frac{\partial \mathcal{L}_{ab}}{\partial \tau}\end{aligned}$$

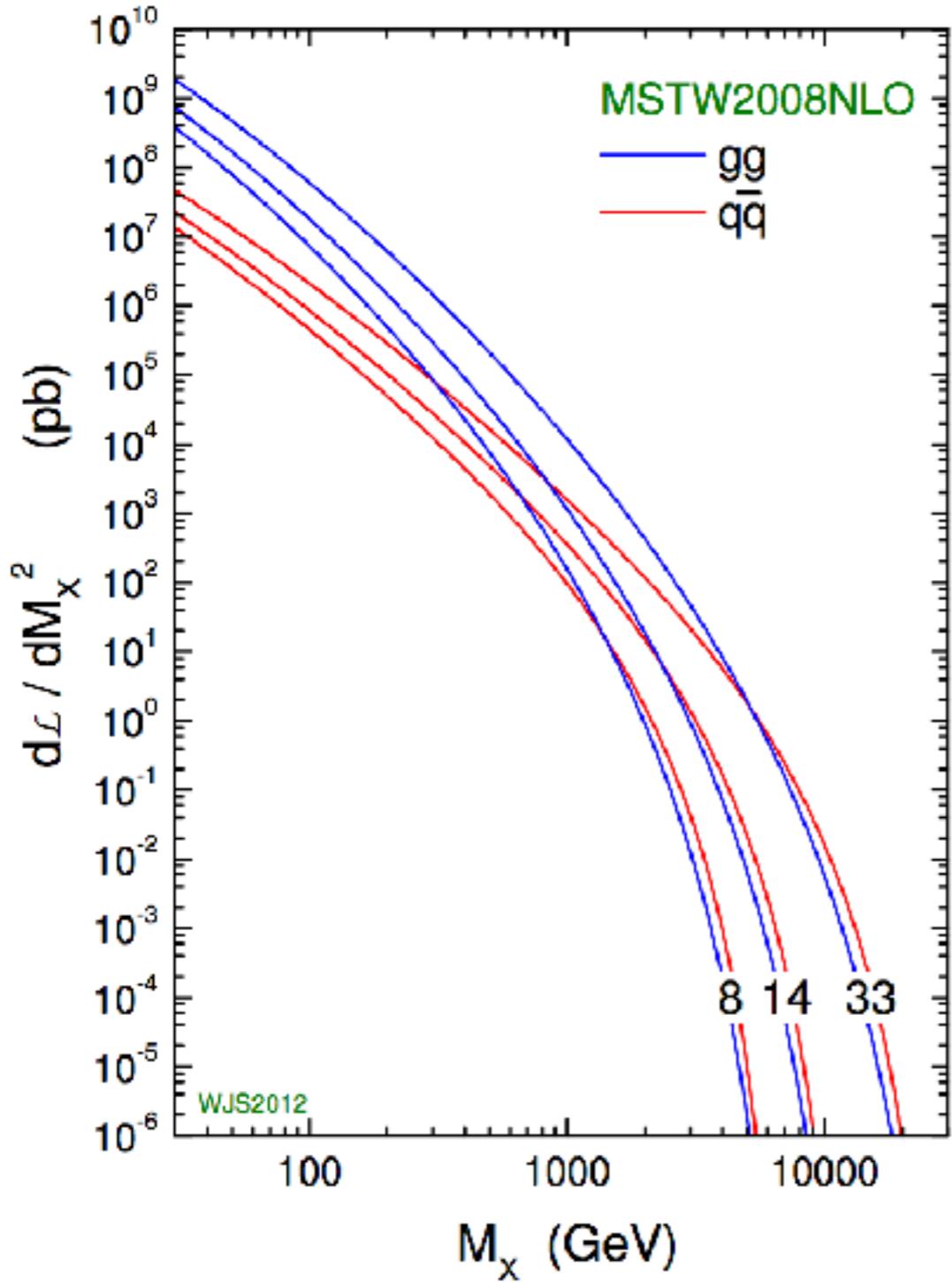
with

$$\tau = \frac{M^2}{\tau}$$

- Define

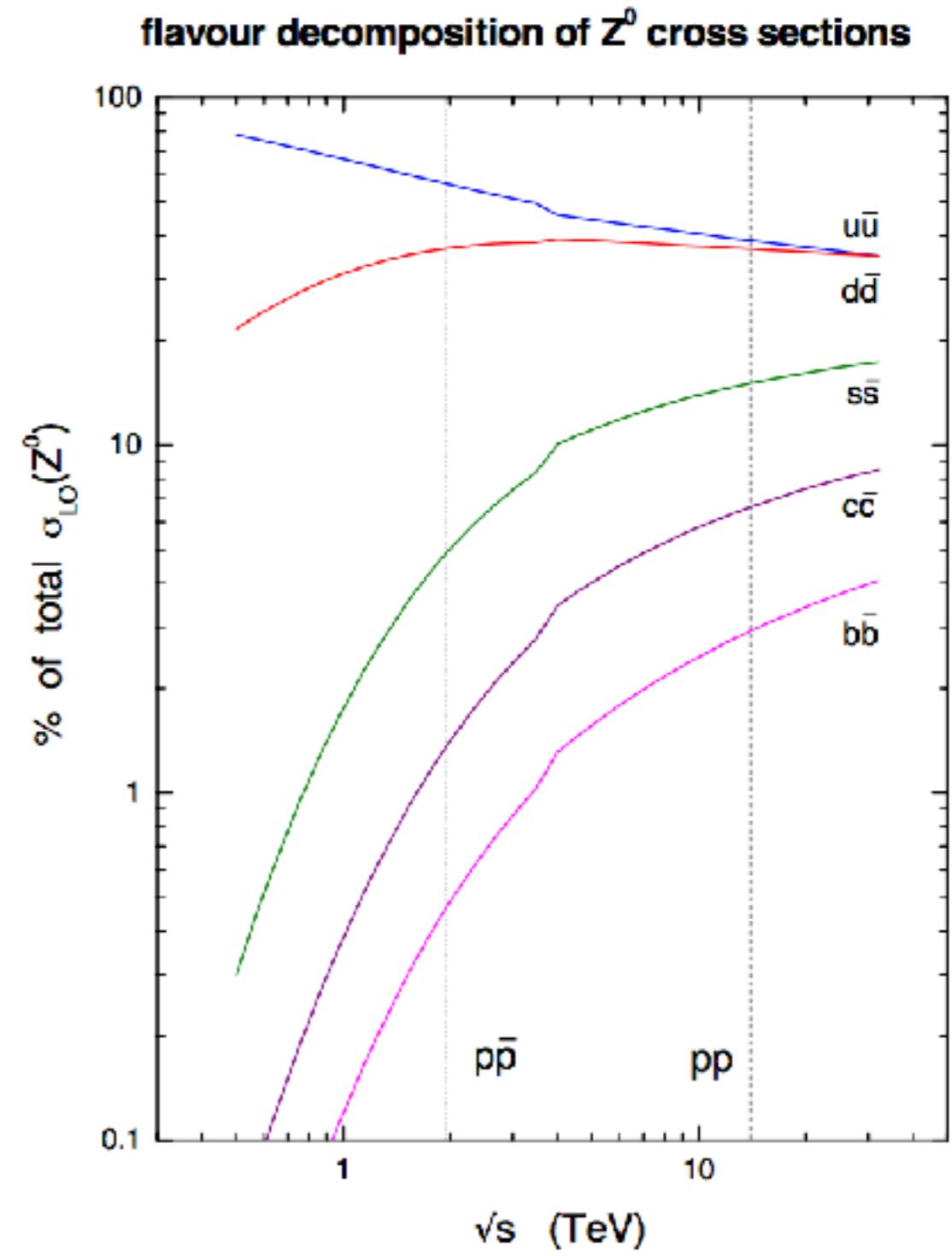
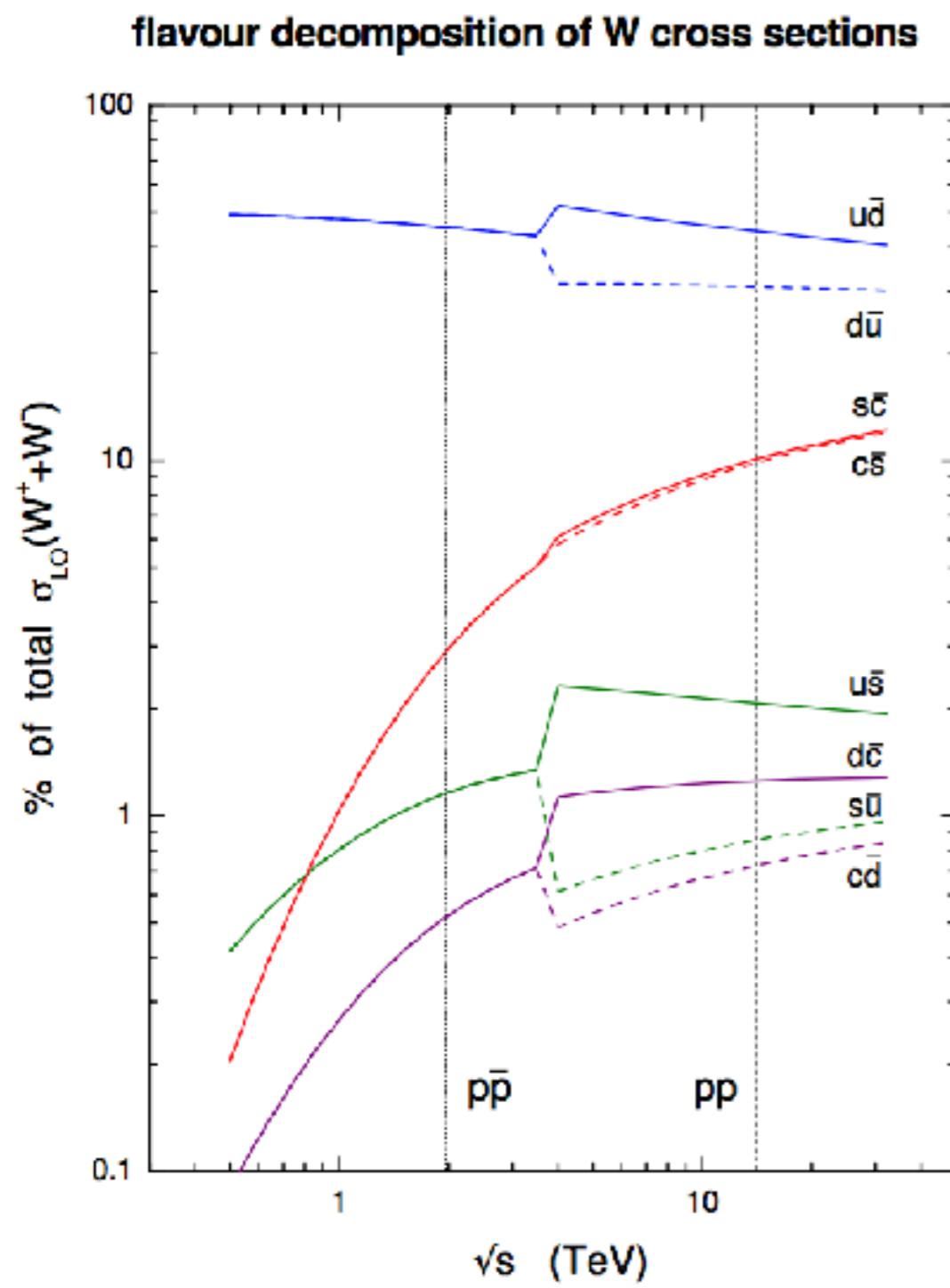
$$\begin{aligned}\Phi_{ab}(M^2) &= \frac{\partial \mathcal{L}_{ab}}{\partial \tau} \\ &= \int_0^1 dx_a dx_b f_a(x_a, M^2) f_b(x_b, M^2) \delta(x_a x_b - \tau) \\ &= \frac{1}{S} \int_\tau^1 \frac{dy}{y} f_a(y, M^2) f_b\left(\frac{\tau}{y}, M^2\right)\end{aligned}$$

# Parton Luminosities



$$\begin{aligned}\Phi_{ab}(M^2) &= \frac{\partial \mathcal{L}_{ab}}{\partial \tau} \\ &= \int_0^1 dx_a dx_b f_a(x_a, M^2) f_b(x_b, M^2) \delta(x_a x_b - \tau) \\ &= \frac{1}{S} \int_\tau^1 \frac{dy}{y} f_a(y, M^2) f_b\left(\frac{\tau}{y}, M^2\right)\end{aligned}$$

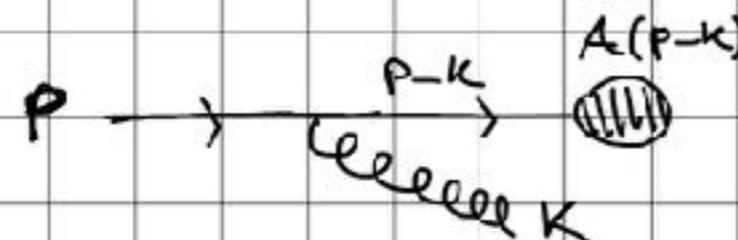
# Parton Luminosities



# Extra material

# QCD and improved parton model

REAL EMISSION



Detailed version

Sudakov parametrization

$$k = (1-z)P + K_T + \xi \eta$$

where  $\eta$  such that  $P \cdot K_T = 0$

$$\eta \cdot K_T = 0$$

$$\eta^2 = 0$$

$$zP \cdot \eta = 1$$

For example

$$P = P^o(1, 0, 0, 1)$$

$$\eta = \frac{1}{4P^o} (1, 0, 0, -1)$$

$$K_T = (0, k_T^1, k_T^2, 0)$$

From  $\eta^2 = 0$  (on-shell condition)

$$\Rightarrow k_T^2 + 2P \cdot \eta \xi (1-z) = 0 \Rightarrow \xi = -\frac{k_T^2}{1-z}$$

From  $(p-\eta)^2 < 0$

$$\Rightarrow z < 1$$

$$\text{From } k_3 = 0 \Rightarrow |k_T^1| < 4P^o(1-z)$$

# QCD and improved parton model

$$\frac{d^3k}{(2\pi)^3 2k^0} = \frac{1}{16\pi^2} \frac{dk_T^2 dz}{(1-z)} \quad (\text{integrating over } d\phi)$$

Singular part of amplitudine:

$$M_{q,sing}^{(1)}(p, u) = q_s A_i(p-u) \frac{z \not{p} - k_T - \cancel{k}_T}{(p-u)^2} \not{\epsilon}(u) \left( \epsilon_{ij}^A \right) U_j(p)$$

$\rightarrow z \not{p} - k_T - \cancel{k}_T$   
 $\rightarrow \frac{1-z}{k_T^2}$   
 $\rightarrow SU(3) \text{ fund. representation}$

$$= \sum_{ij} \frac{(1-z)}{k_T^2} A_i(p-u) (z \not{p} - k_T) \not{\epsilon}(u) \epsilon_{ij}^A U_j(p)$$

$$\text{But } \not{p} U(\not{p}) = 0 \quad \Rightarrow \quad (1-z) \not{p} \not{\epsilon}(u) U(p) = -z \epsilon_u(u) k_T^\mu U(p)$$

$$\Rightarrow M_{q,sing}^{(1)}(p, u) = -\frac{q_s}{k_T^2} A_i(p-u) [z z k_T \epsilon_u(u) + (1-z) k_T \not{\epsilon}(u)] \epsilon_{ij}^A U_j(p)$$

# QCD and improved parton model

$$|M_{q\bar{q}\gamma}^{(1)}|^2 = - \frac{2 g_s^2 C_F}{\kappa_T^2} \frac{(1+z)^2}{z} |M_q^{(0)}(z_P)|^2$$

where we have used

$$\sum \epsilon_\mu(u) \epsilon_\nu^*(u) = -g_{\mu\nu} = -g_{\mu\nu} + \frac{u_\mu u_\nu + u_\nu u_\mu}{u \cdot n}$$

$$\epsilon_{ij}^A \epsilon_{jk}^A = \delta_{ik} C_F$$

$$C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$$

$$\Rightarrow \hat{\sigma}_q^{(1)}(p) = \frac{1}{16\pi^2} \int_0^1 \frac{dz}{1-z} \int_0^{|\kappa_T^z|_{\max}} d|\kappa_T^z| \frac{1}{p \cdot p'} |M_q^{(1)}(p, u)|^2$$

$$ds = \frac{p^2}{4\pi} = \frac{ds}{2\pi} C_F \int_0^1 \frac{dz}{1-z} \int_0^{|\kappa_T^z|_{\max}} d|\kappa_T^z| \frac{1+z^2}{|\kappa_T^z|} \frac{1}{z(p \cdot p')} |M_q^{(0)}(z_P)|^2 + \text{regular terms}$$

$$= \frac{ds}{2\pi} C_F \int_0^1 \frac{dz}{1-z} \int_0^{|\kappa_T^z|_{\max}} d|\kappa_T^z| \frac{1+z^2}{|\kappa_T^z|} \hat{\sigma}_q^{(0)}(z_P)$$

DIVERGENCES!

SOFT

$z \rightarrow 1$

Regulator  $\epsilon \rightarrow 0$

COLLINEAR

$|\kappa_T^z| \rightarrow 0$

Regulator  $\lambda \rightarrow 0$

# QCD and improved parton model

$$\hat{\sigma}_q^{(1)}(p) = \frac{ds}{2\pi} C_F \int_0^{1-\epsilon} \frac{dz}{1-z} \int_{|k_T^z|}^{|k_T^z|_{\max}} \frac{d|k_T^z|}{|k_T^z|} (1+z^2) \hat{\sigma}_q^{(0)}(zp)$$

Adding virtual corrections

$$- \hat{\sigma}_q^{(0)}(p) \frac{ds}{2\pi} C_F \int_0^{1-\epsilon} \frac{dz}{1-z} \int_{z\epsilon}^{|k_T^z|_{\max}} \frac{d|k_T^z|}{|k_T^z|} (1+\epsilon)$$

the soft singularity cancels  
 $\Rightarrow \epsilon \rightarrow 0$

$$\hat{\sigma}_q^{(1)}(p) = \frac{ds}{2\pi} C_F \int_0^1 \frac{dz}{1-z} \int_{z^2}^{|k_T^z|_{\max}} \frac{d|k_T^z|}{|k_T^z|} (1+z^2) [\hat{\sigma}_q^{(0)}(zp) - \hat{\sigma}_q^{(0)}(p)]$$

Still left with COLINEAR divergence!

Introduce  $\mu_F$  to split integration

$$\int_{z^2}^{|k_T^z|_{\max}} \frac{d|k_T^z|}{|k_T^z|} \rightarrow \underbrace{\int_{z^2}^{\mu_F^2} \frac{d|k_T^z|}{|k_T^z|}}_{\text{Singular}} + \underbrace{\int_{\mu_F^2}^{|k_T^z|_{\max}} \frac{d|k_T^z|}{|k_T^z|}}_{\text{finite}}$$

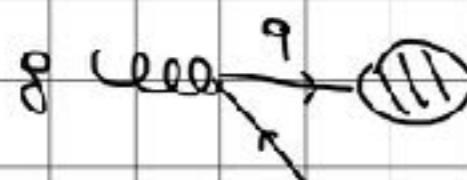
# QCD and improved parton model

$$\Rightarrow \hat{\sigma}_q(p) = \hat{\sigma}_q^{(0)}(p) + \hat{\sigma}_q^{(1)}(p)$$

universal function  $P(q \rightarrow q)$

$$= \hat{\sigma}_q^{(0)}(p) + \frac{g_s}{2\pi} \int_0^1 dz P_{qq}(z) \hat{\sigma}_q^{(0)}(zp) \log \frac{\mu_F^2}{z^2} + \hat{\sigma}_{q,\text{loop}}^{(1)}(p, \mu_F^2)$$

Of course quark can come from gluon



Doing the whole calculation, we get

$$\hat{\sigma}_g(p) = \hat{\sigma}_g^{(0)}(p)$$

$$= \frac{g_s}{2\pi} \int_0^1 dz P_{gg}(z) \hat{\sigma}_q^{(0)}(zp) \log \frac{\mu_F^2}{z^2} + \hat{\sigma}_{g,\text{loop}}^{(1)}(p, \mu_F^2)$$

In the parton model formula

$$\sigma(p) = \int_0^1 dy [f_q(y) \hat{\sigma}_q(y p) + f_g(y) \hat{\sigma}_g(y p)]$$

# QCD and improved parton model

$$\sigma(p) = \int_0^1 dy [f_q(y) \hat{\sigma}_q^{(0)}(y_p) + \frac{\alpha_s}{2\pi} \int_0^1 dy f_q(y) \int_0^1 dz \hat{\sigma}_q^{(0)}(yz_p) P_{qg}(z) \log \frac{\mu_F^2}{z^2} + \frac{\alpha_s}{2\pi} \int_0^1 dy f_g(y) \int_0^1 dz \hat{\sigma}_F^{(0)}(yz_p) P_{qF}(z) \log \frac{\mu_F^2}{z^2} + \int_0^1 dy f_q(y) \hat{\sigma}_{q,reg}^{(1)}(y_p, \mu_F^2) + \int_0^1 dy f_g(y) \hat{\sigma}_{F,reg}^{(1)}(y_p, \mu_F^2)]$$

↓ terms  $\propto \log \frac{\mu_F^2}{z^2}$  can be reabsorbed into redefinition of  $f_g$

$x = yz$

$$f_q(x, \mu_F^2) = \int_x^1 \frac{dy}{y} \left\{ f_q(y) \left[ \delta\left(1 - \frac{x}{y}\right) + \frac{\alpha_s}{2\pi} P_{qg}\left(\frac{x}{y}\right) \log \frac{\mu_F^2}{y^2} \right] + f_g(y) \left[ \frac{\alpha_s}{2\pi} P_{qg}\left(\frac{x}{y}\right) \log \frac{\mu_F^2}{x^2} \right] \right\}$$

# QCD and improved parton model

So that

$$\sigma(p) = \int_0^1 dx f_q(x, \mu_F^\epsilon) \hat{\sigma}_q(xp, \mu_F^\epsilon) + f_p(x) \hat{\sigma}_p(xp, \mu_F^\epsilon)$$



both depend on arbitrary  
FACTORISATION scale

Note that however the dependence of  $f_{q,p}(x, \mu_F^\epsilon)$   
is totally ~~not~~ fixed by perturbation theory

$$\mu^2 \frac{d f_q(x, \mu^2)}{d \mu^2} = \frac{ds}{2\pi} \int_x^1 \frac{dy}{y} \left[ P_{qq} \left( \frac{x}{y} \right) f_q(y, \mu^2) + P_{qF} \left( \frac{x}{y} \right) f_p(y, \mu^2) \right]$$

# Exercise III: PDF evolution

- Consider Z production at Tevatron and at LHC. How to determine the contribution of the different parton channels to the total cross section?
- You might want to use the Parton Luminosity definition

$$\Phi_{ij} = \frac{1}{S_{\text{had}}} \int_{\tau}^1 \frac{dy}{y} f_i(y, M_X^2) f_j \left( \frac{\tau}{y}, M_X^2 \right) \quad \tau = \frac{M_X^2}{S_{\text{had}}}$$

And plot them by using APFELweb or (older) hepdata

<https://apfel.mi.infn.it/>

<http://hepdata.cedar.ac.uk/pdf/pdf3.html>

# HERA data

H1 and ZEUS

