# ~~Estimating~~ Approximating Missing Higher Orders (MHO) in Transverse Momentum Distributions with Resummations

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# Outline

- (Combined) Resummations in a Nutshell
- Approximate N3LO Higgs (pointlike) $p_T$-spectrum
- Towards an approximate N3LO DY $p_T$-spectrum
- Conclusions & Outlook

# Transverse Momentum Distributions

Consider the collision of two protons $p_1 + p_2 \to F(M^2) + X$. Using QCD factorization theorem:

$$\frac{\mathrm{d}\sigma_F}{\mathrm{d}p_T^2}\left(p_T, \frac{M^2}{\mu^2}\right) = \frac{1}{M^2}\sum_{a,b}\int_\tau^1 \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_{ab}\left(\frac{\tau}{x}\right)\frac{\mathrm{d}\hat{\sigma}_{a,b}}{\mathrm{d}p_T^2}\left(x, p_T, \frac{M^2}{\mu^2}\right), \qquad \xi_p \equiv \frac{p_T^2}{M^2}$$

In perturbation theory, the ***partonic*** part is expanded as series in $\alpha_s$:

$$\frac{\mathrm{d}\hat{\sigma}_{a,b}}{\mathrm{d}p_T^2}\left(x, p_T, \frac{M^2}{\mu^2}\right) = \sigma^F_{\text{Born}}\left(\underbrace{\alpha_s \mathcal{C}^{(1)}_{a,b}}_{\text{NLO}} + \underbrace{\alpha_s^2 \mathcal{C}^{(2)}_{a,b}}_{\text{NNLO}} + \underbrace{\alpha_s^3 \mathcal{C}^{(3)}_{a,b}}_{\text{N3LO}} + \cdots\right)\left(x, p_T, \frac{M^2}{\mu^2}\right)$$

**Perturbative computations** <u>assume</u> that the coefficients $\mathcal{C}^{(n)}_{a,b}$ are **WELL-BEHAVED**. But what happens when the smallness of $\alpha_s$ is compensated by **large logarithms** ($\alpha_s^n L^m \sim 1$)?

$$\mathcal{C}^{(n)}_{a,b}(x) = \sum_{m=1}^{2n} c^{a,b}_{m,n}(x) L^m, \qquad L \propto \ln\left(\frac{p_T^2}{M^2}\right), \ln(1-x), \cdots$$

# Resummations in a nutshell

**Conjugate Spaces: (CSS, BCdFG, $\cdots$)**

Mellin space $\leftarrow$ Bypass convolution

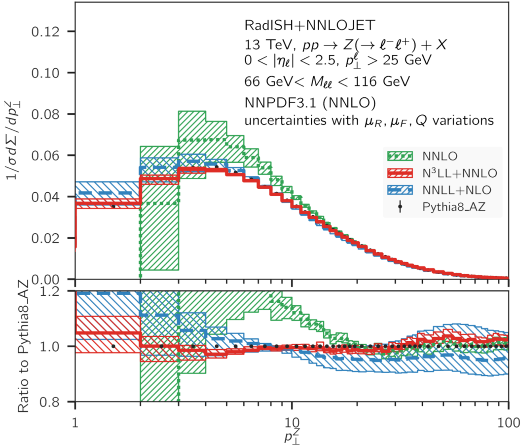
$$\sum_{a,b}\left(\mathcal{L}_{ab}\otimes\frac{\mathrm{d}\hat{\sigma}_{a,b}}{\mathrm{d}p_T^2}\right)(x)\rightarrow\sum_{a,b}\left(\mathcal{L}_{ab}\frac{\mathrm{d}\hat{\sigma}_{a,b}}{\mathrm{d}p_T^2}\right)(N)$$

Fourier space $\leftarrow$ Factorize $\delta$-constraints

$$\int \mathrm{d}^2\vec{p}_T \exp\left(-i\vec{b}\vec{p}_T\right)\delta_{p_T}^{[k]}\rightarrow\prod_{k=1}^{n}\exp\left(-i\vec{b}\vec{p}_{T,k}\right)$$

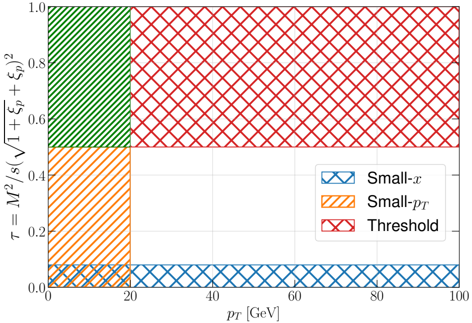
**Direct Spaces: (RadISH)**

$$\frac{\mathrm{d}\hat{\sigma}_{a,b}}{\mathrm{d}p_T^2}=\sigma_{\mathrm{Born}}^F\mathcal{H}(N)\exp\left(\sum_{n=0}^{2n}\alpha_s^{(n-1)}g_n\left(\alpha_s L\right)\right)$$




**[Bizon et al., arXiv:1905.05171, (19)]**

# Resummation regions




| | Direct | Conjugate |
|---|---|---|
| Threshold | $x \to 1$ | $N \to \infty$ |
| Small-$p_T$ | $\xi_p \to 0$ | $b \to \infty$ |
| High-Energy | $x \to 0$ | $N \to 0$ |

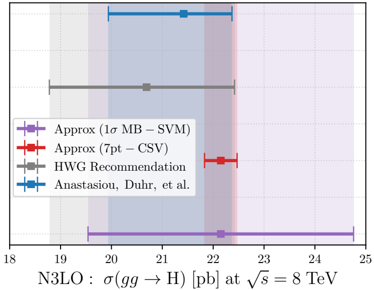
| DIFF \| TOT | Threshold | Small-$p_T$ | High-Energy |
|---|---|---|---|
| Threshold | | NNLL+NNLL$'$ **[CM; TR]** | N3LL+LL$x$ **[SM, MB (18)]** |
| Small-$p_T$ | NNLL+NNLL$'$ **[CM; TR]** | | NNLL+LL$x$ **[SM (15)]** |
| High-Energy | N3LL+LL$x$ **[SM, MB (18)]** | NNLL+LL$x$ **[SM (15)]** | |

# Approximating the next-unknown order with Resummations

**Earlier works in approximating $\sigma\,(gg \to \mathrm{H})$ [RB et al, 1303.3590]:**

$$\sigma^{\mathrm{N3LO}}_{\mathrm{approx}}\left(\tau, m_{\mathrm{H}}^2\right) = \sigma^{\mathrm{Born}}_{gg,H}\left[\sum_{a,b}\left(\delta^{ag}_{bg} + \sum_{n=1}^{2}\alpha_s^n C_{ab}^{(n)}\right)\right] + \alpha_s^3 C^{(3)}_{gg,\mathrm{approx}}$$
$$= (22.41 \pm 0.32)\ \mathrm{pb}, \quad \sqrt{s} = 8\ \mathrm{TeV},$$

for $m_H = \mu_R = \mu_F = 125$ GeV. Approximation constructed by combining the singularity structures at small and large $N$.




**Apply the same approach to Transverse Momentum Distributions:**

$$\frac{d\sigma_H^{(n+1)}}{dp_T^2}\left(N, \frac{p_T^2}{M^2}\right) = \frac{d\sigma_H^{(n)}}{dp_T^2}\left(N, \frac{p_T^2}{M^2}\right) + \alpha_s^{(n+1)}\frac{d\sigma^{[n+1]}_{\mathrm{H,approx}}}{dp_T^2}\left(N, \frac{p_T^2}{M^2}\right)$$

$$\boxed{\frac{d\sigma^{[n+1]}_{\mathrm{H,approx}}}{dp_T^2}\left(N, \frac{p_T^2}{M^2}\right) = \frac{d\sigma^{[n+1]}_{\mathrm{H,TH}}}{dp_T^2}\left(N, \frac{p_T^2}{M^2}\right) + \frac{d\sigma^{[n+1]}_{\mathrm{H,HE}}}{dp_T^2}\left(N, \frac{p_T^2}{M^2}\right)} \quad (1)$$

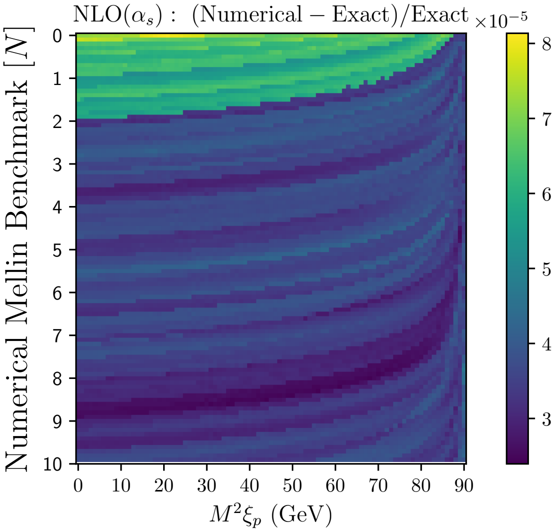
# Approximating the next-unknown order with Resummations

## Key steps/ingredients:

- **Mellin** space Resummed expressions to interpolate between the various kinematic limits
- Modify the Resummed expressions to get rid of **spurious singularities**
- Validate the methodology against the exact FO computations ⟵ requires a Mellin space version of the full FO:

$$\int_0^1 \mathrm{d}x\, x^{N-1} \int \mathrm{dy} \frac{\mathrm{d}\sigma_{\mathrm{H}}}{\mathrm{d}p_T^2 dy}$$

- A more reliable way to estimate the resulting **uncertainties** due to the approximation

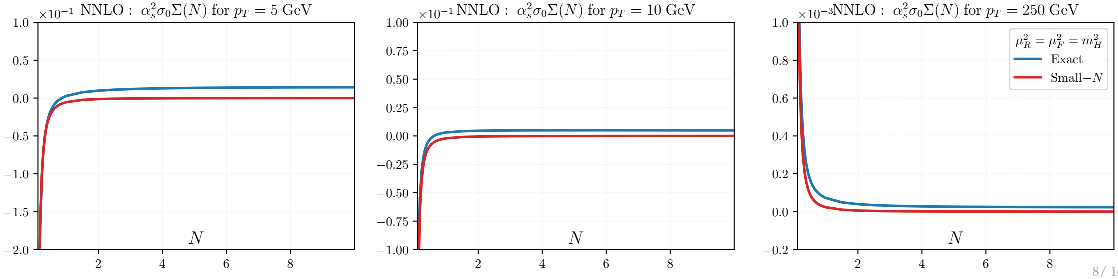


# Validation @ NNLO: Approximation from High-Energy (Small-$x$ / $N$)

The LL$x$ HE resummation of the Higgs double differential cross-section is given by **[SF & CM, 1511.05561]**:

$$\frac{d\sigma^{H,\mathrm{LL}x}_{gg,\mathrm{HE}}}{dy\,dp_T^2}\left(\tilde{\gamma}^{\pm} \equiv \gamma\left(N \pm i\frac{bM}{2}\right), \xi_p\right) = \sigma^{\mathrm{Born}}_{gg,H}\frac{\xi_p^{\tilde{\gamma}^+}\xi_p^{\tilde{\gamma}^-}}{\xi_p(1+\xi_p)^N} R\left(\tilde{\gamma}^+\right) R\left(\tilde{\gamma}^-\right) \times$$

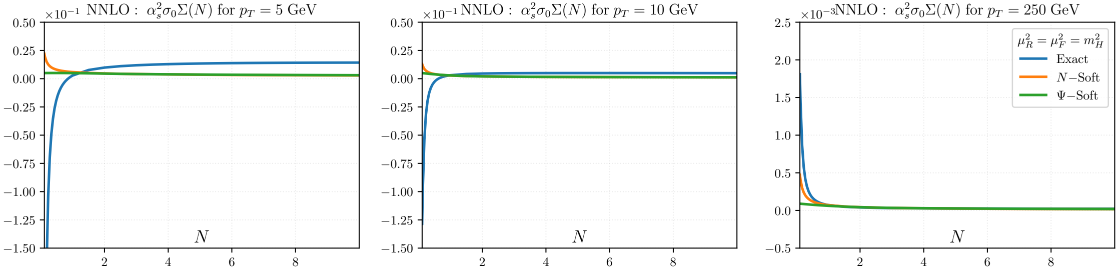
$$0 \xleftarrow{N\longrightarrow\infty} \left[\left(1+\frac{2\tilde{\gamma}^+\tilde{\gamma}^-}{1-\tilde{\gamma}^+-\tilde{\gamma}^-}\right)\frac{\Gamma\left(1+\tilde{\gamma}^+\right)\Gamma\left(1+\tilde{\gamma}^-\right)\Gamma\left(2-\tilde{\gamma}^+-\tilde{\gamma}^-\right)}{\Gamma\left(2-\tilde{\gamma}^+\right)\Gamma\left(2-\tilde{\gamma}^-\right)\Gamma\left(\tilde{\gamma}^++\tilde{\gamma}^-\right)}\right]$$



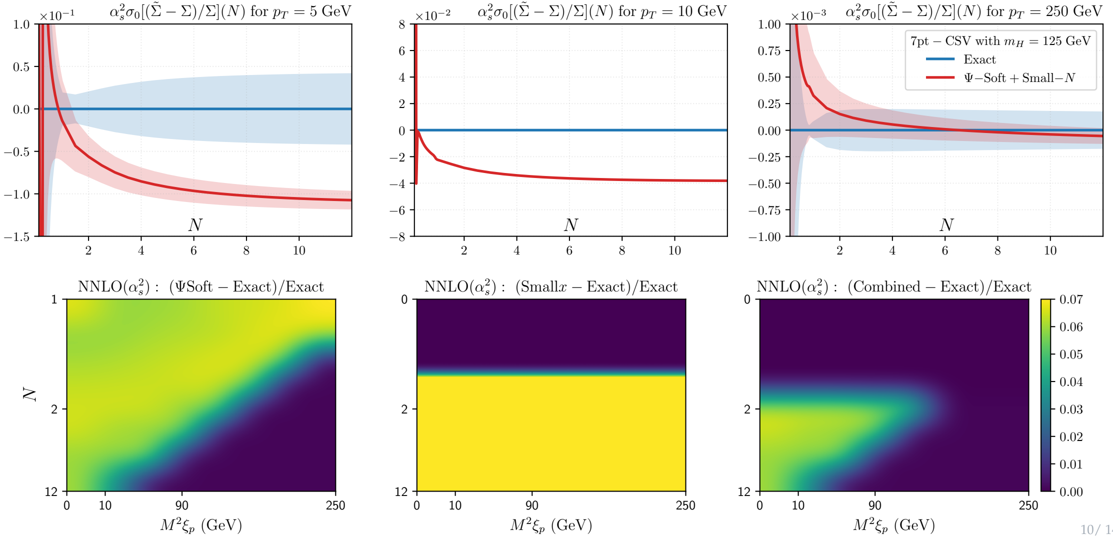
# Validation @ NNLO: Approximation from High-Energy (Large-$x/N$)

The approximation from the threshold resummation yields **incorrect** singularity structures at small $N$

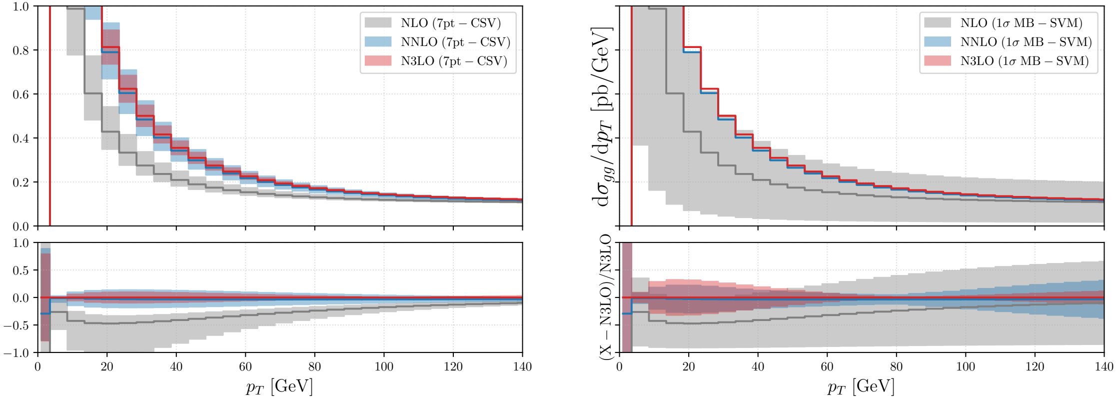
| | | | |
|---|---|---|---|
| Resummation | $\ln(N)$ | $\xrightarrow{\mathcal{M}^{-1}[\,]}$ | $\left[\frac{\ln(\ln 1/x)}{\ln 1/x}\right]_+$ |
| Exact FO | $\left(\frac{\gamma_E^2+\zeta_2}{2}\right) + \gamma_E \Psi_0(N) + \left(\frac{\Psi_0^2(N) - \Psi_1(N)}{2}\right)$ | $\xleftarrow{\mathcal{M}\ [\,]}$ | $\left[\frac{\ln(1-x)}{(1-x)}\right]_+$ |



# Validation @ NNLO: Small-$N$+$\Psi$-Soft Approximation



# N3LO Approximation @ LHC




**Good perturbative convergence: only an increase of 2-5% wrt NNLO** for the central value
(No correlations between the bins in the MB-SVM)

# Public codes are available

**HPT-MON**: github/N3PDF/HpT-MON

Stands for Higgs $p_T$ Distribution in Momentum and N space. It computes the partonic and hadronic Higgs cross sections from a gluon fusion (in $pp$ collision) up to NNLO both in the momentum $x$ and Mellin $N$ space.

**HPT-N3LO**: github/N3PDF/HpT-N3LO

Implements the expansion of the small- and large-$x$ resummation. Interfaced with HPT-MON, it approximates the N3LO Higgs $p_T$ distribution by constructing the extra-order with the consistent matching of the two resummations.

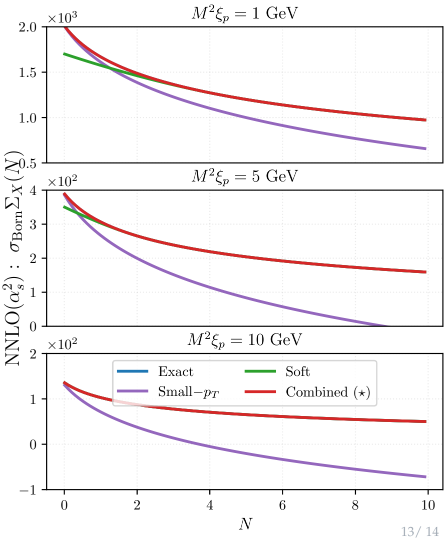
# Towards N3LO DY Transverse Momentum Distributions (Preliminary)

- Mellin space version of the full FO computations already available up to NNLO
- Both the expansions of the resummed threshold (small-$x$ or small-$N$) and small-$p_T$ expressions already available (both from NNLL resummed expressions)
- The threshold and small-$p_T$ expressions are combined using a profile matching

$$\frac{d\sigma^{[m]}_{\mathrm{DY},\star}}{dp_T^2} = T(N,\xi_p)\frac{d\sigma^{[m]}_{\mathrm{DY,TH}}}{dp_T^2} + (1 - T(N,\xi_p))\frac{d\sigma^{[m]}_{\mathrm{DY},p_T}}{dp_T^2}$$

where $T(N,\xi_p)$ interpolates between the soft and small-$p_T$ approximation in the respective limits

- Analytical expression of the high-energy resummed expression only available in the large-$b$ (small-$p_T$) limit **[SM (16)]** Non-trivial given the relation between $\ln(N)$ and $\ln(p_T)$



**Conclusions**:

- Resummed predictions contain at all-orders contributions for given kinematic limits
- Approximate expression for N3LO pointlike Higgs transverse momentum distribution constructed by combining threshold and high-energy resummations
- Combined resummed expression provides a potential tool to approximate Missing Higher Orders in perturbative computations
- Codes are publicly available for the Higgs transverse momentum distributions

**Outlook**:

- Complete the N3LO approximation for the DY transverse momentum distributions: requires the derivation of the high-energy resummed expression at finite $p_T$
- Extend the formalism to various differential distributions (rapidity, invariant mass, etc.)

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## Outlook:

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**THANK YOU FOR YOUR ATTENTION**

# Backup

# Estimating MHOU in Perturbation Theory

Transverse momentum distributions are computed in perturbation theory as:

$$\frac{\mathrm{d}\sigma_F}{\mathrm{d}p_T^2}(p_T) \simeq \sum_{k=m}^{n} \alpha_s^k \mathcal{C}_k(p_T) + \mathcal{O}\left(\alpha_s^{n+1}\right)$$

The peturbative expansions are ***asymptotic*** to $(\mathrm{d}\sigma_F/\mathrm{d}p_T^2)$ i.e. (up to some order) increasing in powers of $\alpha_s$ improves the series approximation.

$$\frac{\mathrm{d}\sigma_F}{\mathrm{d}p_T^2}(p_T) \simeq \frac{\mathrm{d}\sigma_F^{(n)}}{\mathrm{d}p_T^2}(p_T) + \Delta_{\mathrm{MHO}}$$

How to estimate $\Delta_{\mathrm{MHO}}$?

Renormalization in QFT introduces an unphysical dependence $\mu$. Despite the fact that RGE states that physical observables are independent of $\mu$ ($\mu\partial\mathrm{d}\sigma/\partial\mu\mathrm{d}p_T^2 = 0$), residual $\mu$-dependence appear in perturbative computations.

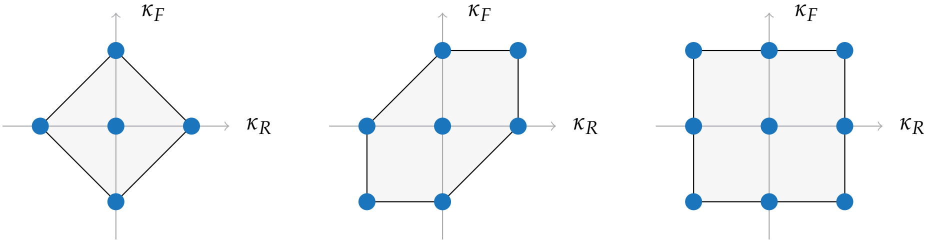
$$\mu\frac{\partial}{\partial\mu}\left(\frac{\mathrm{d}\sigma_F^{(n)}}{\mathrm{d}p_T^2}\right) = \mathcal{O}\left(\alpha_s^{n+1}\right) = \mathcal{O}\left(\Delta_{\mathrm{MHO}}\right)$$

# Canonical Scale Variation (CSV)

**CANONICAL METHOD:** Variation by a factor of 2 around a **central scale** $\mu_0$.

$$\frac{\mathrm{d}\sigma_F}{\mathrm{d}p_T^2} \simeq \frac{\mathrm{d}\sigma_F^{(n)}}{\mathrm{d}p_T^2}(\mu_0) \pm \max_{\substack{\mu_{\min}\leq 2\mu_0 \\ 2\mu_{\max}\geq\mu_0}} \left| \frac{\mathrm{d}\sigma_F^{(n)}}{\mathrm{d}p_T^2}(\mu) - \frac{\mathrm{d}\sigma_F^{(n)}}{\mathrm{d}p_T^2}(\mu_0) \right|$$

For a multi-scale process involving the ***renormalization scale*** $(\mu_R = \kappa_R\mu_0)$ and the ***factorization scale*** $(\mu_F = \kappa_F\mu_0)$, there exists various prescriptions:



# Pros & Cons of CSV

**Advantages**

- **Renormalization Group Invariance** ensures that $\mu$-dependence decrease with increasing order
- Lead to smooth functions, incorporating correlations between nearby regions in the Phase Space
- Universal and therefore can be applied to any processes

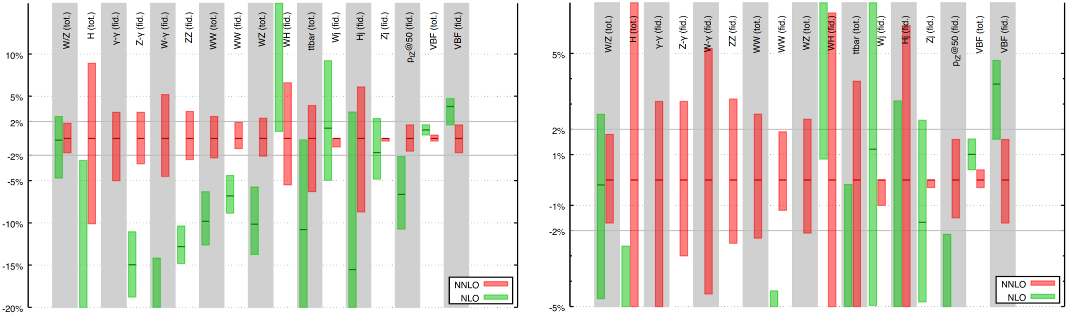
**Caveats**

- Lack of Probabilistic Interpretation (impossibility of assessing the degree of belief)
- Ambiguity in defining the central scale and the ranges at which the scales should vary
- Do not account for singularities that appear at higher orders

**Alternatives**

- **Cacciari-Houdeau:** uses a Bayesian model to infer on the *hidden parameters* that are assumed to bound the structure of the perturbative coefficients **[MC, NH, 1105.5152]**
- **Bonvini's models:** built upon Cacciari-Houdeau's work to construct more general models while addressing its limitations (Geometric Model [GM], Scale Variation Model [SVM], $\cdots$) **[MB, 2006.16293]**

# How good is the CSV method?




NNLO predictions just barely reach **1%** and for many processes the scale band is $\sim \pm$**2%**
Only **3/17** cases in which the NNLO central values are contained in NLO uncertainty band