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Asymmetric soft resummation of Semi-Inclusive Deep Inelastic scattering

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Introduction

Decades of research aimed at understanding fundamental physics have led to the development of the so-called Standard Model, which is the theory of fundamental interactions. This framework successfully describes three out of the four fundamental forces: electromagnetic, weak, and strong interactions. The theory has been extensively validated through experiments conducted at particle accelerators. In these experiments, different types of particles (leptons or hadrons) are accelerated and collided, and one or more of the resulting final states are measured. This measurement is carried out by evaluating an observable; in particle physics, the primary observables are the total and differential cross sections of scattering processes. In the framework of Quantum Field Theory, cross sections are determined through a perturbative expansion, where the calculation of each successive order improves the precision of the predictions.

The study of processes involving identified final hadrons states plays a crucial role in Quantum Chromodynamics (QCD). Such processes enable the exploration of hadronic structure and its underlying dynamics. In this thesis, I focus on Semi-Inclusive Deep Inelastic Scattering (SIDIS)namely the process where a hadron interacts with a lepton producing a lepton, an identified hadron, and a generic hadronic state, namely $h_1, l \to h_2, l, X$, where X denotes the generic final state. SIDIS process compared to the inclusive case: the Deep Inelastic Scattering (DIS), also provides, in addition to the detection of the final lepton state, the detection of one of the final hadron state. In the framework of perturbative QCD and since the factorization theorem, the SIDIS cross section is expressed as a convolution of non-perturbative and universal Parton Distribution Functions (PDFs) and Fragmentation Functions (FFs) with the perturbative, process-dependent partonic cross section, also referred to as the coefficient function. Therefore, one of the main applications of SIDIS processes is the extraction of PDFs and FFs, which provide crucial insights into the internal structure of the nucleon. However, unlike PDFs, for FFs there is limited data available for their determination [1]. This scarcity makes the SIDIS reaction an important area of study to expand the existing dataset.

Furthermore, SIDIS will be one of the main processes studied at the Electron-Ion Collider (EIC), which will exploit high-energy electrons and ion beams to simultaneously extract FFs and PDFs. In particular at EIC, SIDIS measurements with polarized beams and/or targets will provide valuable insights into the spin distribution of nucleons, hence on their spin-dependent PDFs across a wide range of energy scales.

In light of the above considerations, a high-precision evaluation of the perturbative corrections to the SIDIS coefficient function is crucial for extracting theoretical predictions for future experiments. Now, at the parton level the SIDIS process is viewed as $p + \gamma^* \to p' + X$, where γ^* is the virtual photon which mediates the interaction with the incoming lepton, while p is the incoming parton and p' is the outgoing parton which fragments in the final measured hadronic state. In this context, coefficient functions (or equivalently the partonic cross sections) are typically expressed in terms of the hard scale $Q^2 = -q^2$ and the scaling variables $\hat{x} = Q^2/2(p_1 \cdot q)$ and $\hat{z} = p_1 \cdot p_2/p_1 \cdot q$, with p_1, p_2 and q are the momenta of the incoming parton, outgoing parton and the virtual gauge boson respectively. In particular, the coefficient function is differential in both $\hat{x}$ and $\hat{z}$ variables. For the quark-to-quark channel, the cancellations between the real gluon emissions and virtual corrections produce terms of the type $\alpha_s^k\delta(1-\hat{x})\left(\frac{\ln^m(1-\hat{z})}{1-\hat{z}}\right)_+$, $\alpha_s^k\delta(1-\hat{z})\left(\frac{\ln^m(1-\hat{x})}{1-\hat{x}}\right)_+$ with $m\leq 2k-1$ and "mixed" terms $\alpha_s^k\left(\frac{\ln^m(1-\hat{x})}{1-\hat{x}}\right)_+\left(\frac{\ln^n(1-\hat{x})}{1-\hat{x}}\right)_+$ with $m+n\leq 2k-2$, where the symbol $+$ denote the plus-distribution. The aim of this thesis is to study the so-called threshold limit, i.e. the regime where the real gluon emission is suppressed, becoming soft. This corresponds to two kinematical configurations:

- The **double-soft limit** corresponds to the elastic configuration, i.e. there is no momentum exchange between the incoming and outgoing partons. In this limit, the radiation emitted by both partons becomes soft, i.e. $\hat{x},\hat{z}\to 1$. In Mellin space, this behaviour translates to the conjugate variables approaching infinity, $N,M\to\infty$.

- The **single-soft limit**, also referred to as the asymmetric case, occurs when either the incoming or outgoing parton carries a fixed longitudinal momentum and the transferred momentum approaches its minimum value to allow this configuration. In these limits the radiation emitted with respect to either the incoming or outgoing parton becomes soft, implying either $\hat{x}\to 1$ or $\hat{z}\to 1$. In terms of the conjugate Mellin variables, this corresponds to either $N\to\infty$ or $M\to\infty$.

Then in these configuration the aforementioned logarithms become enhanced to all orders spoiling the perturbative approach. This problem is addressed using the *threshold resummation* approach. As has been known for a long time [2], threshold resummation leads to an exponentiation of soft logarithms in Mellin space to all orders in the strong coupling constant. Therefore, the main goal is to determine the coefficients of the exponentiation by comparing the expansion of the resummation formula up to a fixed order in α_s with the corresponding fixed-order result.

To clarify, note that the expansion of the resummation formula organizes the soft logarithms into towers of logarithms. For instance, in the double-soft limit in Mellin space, the logarithmic terms in x,z-space correspond to terms of the form $\alpha_s^k(\ln(N)+\ln(M))^n$. At first order in α_s, a comparison with the fixed-order result at NLO allows us to determine the coefficients for the leading logarithms (LL) and next-to-leading logarithms (NLL). These coefficients generate, to all orders, terms of the following types: at LL, only $n=2k$; at NLL, $n=2k$, $2k-1$, $2k-2$; at NNLL, $n=2k$, $2k-1$, $2k-2$, $2k-3$, $2k-4$; and so on. Whereas in the single-soft limits are also included all those logarithmic terms that are power suppressed by some powers of either the variable N or M.

In particular, from the study of the phase space limit, one can observe that the behaviour of the SIDIS coefficient function in the soft limits is completely equivalent to that obtained for the Drell-Yan (DY) process at fixed rapidity [3]. Specifically, the Drell-Yan process can be viewed as the crossed version of the SIDIS case, namely the process where two hadrons collide producing a lepton pair, or equivalently, at the parton level, the interaction of two partons producing a gauge boson, that at fixed rapidity conditions has its longitudinal momentum fixed. In this case, the double-soft limit corresponds to the situation where the gauge boson is produced at rest, while the single-soft limits correspond to the case where the gauge boson has a fixed longitudinal momentum and the energy approaches its minimum value to allow this configuration. Then, using the correspondence between the DY and SIDIS process it is possible to obtain the resummation formula for SIDIS case directly from the one obtained for the DY case in [3]. In particular this approach provides theoretical predictions on the behaviour of the resummation coefficients for SIDIS case.

In this thesis, I study both the single- and double-soft limits for SIDIS process, obtaining resummation coefficients up to NNLL accuracy. Whereas the double-soft limit has already known up to N^{3}LL [4] the single-soft limit is a completely novel result provided by this thesis.

This thesis is structured as follows. In Chapter 1, we provide an overview of the perturbative QCD approach to the study of strong interactions. In particular, we introduce the factorization theorem and the PDFs. We then analyze the DIS process, focusing on the appearance of collinear and soft singularities and how they can be handled in order to obtain finite results. Finally, we discuss the emergence of soft logarithms and the DGLAP equation. In particular, in this chpater we provide the notation that is used for the rest of the thesis. In Chapter 2, we study the SIDIS kinematics and its phase space in the threshold limit, highlighting its correspondence with the DY process. In Chapter 3 we obtain the resummation formula for a process with a single soft scale dependence through a renormalization group argument, then providing the resummation formula for SIDIS case. In Chapter 4, we provide the theoretical predictions for the SIDIS case using the results obtained for the DY process in [3]. We then present the calculations performed to derive the resummation coefficients. Finally, we present the results for both double- and single-soft limits

Contents

Chapter 1

Perturbative QCD

In this chapter, we provide an overview of some of the most important basic concepts of the theory of Quantum Chromodynamics (QCD) in the context of the perturbative regime (pQCD). In particular, we address the key concepts required to understand the results presented in chapters 3 and 4, which constitute the main results of this thesis. We start with a short introduction about the Lagrangian that governs QCD dynamics, and we introduce the renormalization group equation with a focus on the strong interaction coupling constant (α_s).

We introduce the factorization, which allows us to study QCD in the perturbative regime, and we review one of the most important processes in QCD: Deep Inelastic Scattering (DIS). In this way, we can introduce the soft and collinear singularities which arise in perturbative calculations in the context of the parton model. Therefore, we introduce the Altarelli–Parisi equations (GLAP equations), highlighting their properties.

Thanks to this introduction, we can introduce the notation that we use for the rest of this thesis. The interested reader can find more details on these basic arguments in many textbooks; in this chapter, we follow [5], [6], [7] and [8]. For a detailed overview with a modern approach to the DIS process one can see [9].

1.1 Basics of QCD

Quantum Chromodynamics (QCD) is the fundamental theory that governs the strong interaction, describing how quarks and gluons interact, and consequently explaining the forces between nucleons. As a key component of the Standard Model, QCD is formulated as a quantum field theory, specifically a Yang-Mills theory with an $SU(3)$ gauge symmetry. Specifically, the colour charge represents the internal degree of freedom responsible for the strong interaction, and it is characterized by three independent components. The fermion field ψ_q (the so-called quark field) is described by a triplet in this colour space, then its transformations can be represented through the generators of $SU(3)$. These generators in matrices space are 3×3 unitary matrix denoted by the symbol T^a. These matrices generate the rotation in the 3-dimensional complex colour space. Then, the transformation of the quark field (under which the Lagrangian of the strong interaction must be invariant) can be written as the following local transformation

$$\psi_i \to \psi_i' = \exp\left[i\theta^a(x)T^a\right]\psi_i(x)\,, \tag{1.1.1}$$

where $\theta^a(x)$ is an arbitrary function. Hence, the Lagrangian that is invariant under these rotations takes the standard form characteristic of gauge theories, and is given by:

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}_i\left(i\gamma^\mu(D_\mu)_{ij} - m\,\delta_{ij}\right)\psi_j - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a\,, \tag{1.1.2}$$

where

- $(D_\mu)_{ij} = \partial_\mu \delta_{ij} - ig\,(T_a)_{ij}\,\mathcal{A}^a_\mu$: this is the gauge covariant derivative, which couples the quark fields to the gluon fields via the strong coupling constant g. Here, $\mathcal{A}^a_\mu$ denotes the gluon fields, with a being the colour index running from 1 to 8, and μ the space-time index.
- $G^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + g f^{abc} \mathcal{A}^b_\mu \mathcal{A}^c_\nu$, is the generalized Faraday tensor, where f^{abc} are the structure constants of the gauge group, defined through the commutation relation $[T_a, T_b] = i f^{abc} T_c$. This tensor reduces to the electromagnetic field strength tensor in the Abelian case, where the group is commutative, there is only a single charge, the gluon colour indices disappear, and the term involving the structure constants vanishes.

Due to the gauge symmetry of QCD, the expression for the gluon two-point function (i.e., the propagator) derived from the gauge-fixed Yang-Mills Lagrangian is not unique, similarly to what happens in QED. This non-uniqueness introduces an ambiguity parametrized by a gauge parameter, which can be freely chosen. In non-Abelian gauge theories, the Lagrangian additionally includes two extra terms: a gauge-fixing term and a term involving ghost fields, typically denoted by c. The role of the ghost fields is to cancel the contributions from the unphysical gluon states, specifically those corresponding to time-like and longitudinal polarizations. When these additional terms are incorporated into the Lagrangian in 1.1.2, it attains its complete form.

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \sum_{f=q,\bar{q}} \bar{\psi}_f (i\gamma^\mu D_\mu - m_f)\psi_f - \frac{1}{2\xi}(\partial^\mu \mathcal{A}^a_\mu)^2 - \bar{c}^a \partial^\mu D^{ab}_\mu c^b \tag{1.1.3}$$

In particular, the ghost fields arise from the quantization of gauge theories via the path-integral formalism (specifically, through the Faddeev-Popov procedure). This approach also leads to the derivation of the Feynman rules for the theory.

The Lagrangian depends on seven parameters: the masses m_q of the quarks and the coupling constant g_s. It is customary to express the dependence on g_s through the so-called QCD coupling constant α_s, defined as

$$\alpha_s = \frac{g_s^2}{4\pi}\,. \tag{1.1.4}$$

Whereas, for the quark masses and their electric charges one can see the Tab. 1.1.

flavour	Up (u)	Down (d)	Charm (c)	Strange (s)	Top (t)	Bottom (b)
Electric Charge	$+2/3$	$-1/3$	$+2/3$	$-1/3$	$+2/3$	$-1/3$
Mass	~ 2.2 Mev	~ 4.7 Mev	~ 1.3 Gev	~ 0.1 Gev	~ 173 Gev	~ 4.18 Gev

Table 1.1: Properties of the six quarks: electric charge and mass.

1.1.1 Renormalization Group Equation

To introduce the concept of the running coupling, let us consider a dimensionless physical observable $\widehat{O}$ that depends on a single energy scale Q. We assume that Q is much larger than any other dimensionful parameter in the theory, such as quark masses, allowing us to neglect all masses and effectively treat them as zero. Under naive scaling arguments, one might expect that, since $\widehat{O}$ depends only on a single large scale, its value should remain constant and independent of Q. However, this expectation does not hold in a renormalizable quantum field theory. When $\widehat{O}$ is computed as a perturbative expansion in the coupling constant $\alpha_s = g_s^2/(4\pi)$, ultraviolet divergences arise from loop diagrams. The removal of these divergences through renormalization introduces an additional mass scale μ_{R}. Consequently, the observable $\widehat{O}$ generally depends on the ratio Q^2/μ_{R}^2. Moreover, the coupling constant $\alpha_s(\mu_{\text{R}})$ itself acquires a dependence on the renormalization scale μ_{R}.

However, μ is an arbitrary parameter and does not explicitly appear in the QCD Lagrangian, although its choice is necessary to properly define the theory at the quantum level. Therefore, if we hold the bare coupling fixed, physical quantities such as $\widehat{O}$ cannot depend on the choice made for μ_{R}. Additionally, since $\widehat{O}$ is dimensionless, while both Q and μ_{R} carry dimensions of energy, it follows that $\widehat{O}$ can only depend on the dimensionless ratio Q^2/μ_{R}^2 and on the coupling constant α_s. These considerations lead to the conclusion that the observable $\widehat{O}$ satisfies the following differential equation:

$$\mu_{\mathrm{R}}^2 \frac{d}{d\mu_{\mathrm{R}}^2} \widehat{O}\left(\frac{Q^2}{\mu_{\mathrm{R}}^2}, \alpha_s(\mu_{\mathrm{R}}^2)\right) = \left[\mu_{\mathrm{R}}^2 \frac{d}{d\mu_{\mathrm{R}}^2} + \mu_{\mathrm{R}}^2 \frac{\partial \alpha_s(\mu_{\mathrm{R}}^2)}{\partial \mu_{\mathrm{R}}^2} \frac{\partial}{\partial \alpha_s}\right] \widehat{O} = 0 \,. \tag{1.1.5}$$

Thus we define the so-called β function as follows

$$\beta(\alpha_s(\mu_{\mathrm{R}}^2)) \equiv \mu_{\mathrm{R}}^2 \frac{\partial \alpha_s(\mu_{\mathrm{R}}^2)}{\partial \mu_{\mathrm{R}}^2} \,, \tag{1.1.6}$$

and t as

$$t \equiv \ln \frac{Q^2}{\mu_{\mathrm{R}}^2} \,, \tag{1.1.7}$$

so we rewrite the above equation as follows

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right] \widehat{O}\left(e^t, \alpha_s(\mu_{\mathrm{R}}^2)\right) = 0 \,. \tag{1.1.8}$$

What we have just derived represents a specific case of a more general and fundamental equation in quantum field theory, known as the Callan-Symanzik equation.

In order to solve the last equation, we need to introduce the running coupling $\alpha_s(Q)^2$, which is defined implicitly as follows

$$t = \int_{\alpha_s}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)} \qquad \alpha_s = \alpha_s(\mu_{\mathrm{R}}^2) \,. \tag{1.1.9}$$

By differentiating this last equation we note that

$$\beta(\alpha_s(Q^2)) = \frac{\partial \alpha_s(Q^2)}{\partial t} \,, \tag{1.1.10}$$

Thanks to these ingredients, it becomes straightforward to construct a general solution of Eq.1.1.8, namely $\widehat{O}(1, \alpha_s(Q^2))$. The analysis developed so far highlights that all the scale dependence of $\widehat{O}$ is encoded in the running of the coupling constant $\alpha_s(Q^2)$. Therefore, in order to predict $\widehat{O}(1, \alpha_s(Q^2))$, it is essential to solve Eq.1.1.9.

In QCD, the β function has the perturbative expansion

$$\beta(\alpha_s) = -\beta_0 \alpha_s^2 (1 + \beta_1 \alpha_s + \beta_2 \alpha_s^2 + \dots) \tag{1.1.11}$$

where the β_i coefficients are known up to 4 loops (i = 3) (see Appendix B). The leading coefficient is

$$\beta_0 = \frac{11 C_A - 2 N_f}{12\pi} \tag{1.1.12}$$

where $C_A = 3$ and N_f is the number of active flavours, then the numbers of fermion of the theory. We note that $\beta_0 > 0$ as long as $N_f < 17$. If we solve the equation 1.1.9 up to order α_s^2, we obtain

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu_{\mathrm{R}}^2)}{1 + \beta_0 \alpha_s(\mu_{\mathrm{R}}^2) \ln \frac{Q^2}{\mu_{\mathrm{R}}^2}} \tag{1.1.13}$$

which give the relation between $\alpha_s(Q^2)$ and $\alpha_s(\mu_R^2)$ if both are in the perturbative region. Therefore, if $Q^2/\mu_R^2 \to \infty$, then $\alpha_s(Q^2) \to 0$, this property takes the name of asymptotic freedom. The latter is a peculiar property of the QCD, which causes interactions between particles to become asymptotically weaker as the energy scale increases (or equivalently the corresponding length scale decreases). We remark that the sign of β_0 is crucial, because if this were the opposite we should have observed an increase in the coupling constant α_s as the scale Q grows (same as in QED).

Lastly, we define Λ_{QCD} as the energy scale which satisfies the identity:

$$1 + \beta_0 \alpha_s(\mu_R^2) \ln \frac{\Lambda_{QCD}^2}{\mu_R^2} = 0\,, \tag{1.1.14}$$

then the running coupling becomes

$$\alpha_s(Q^2) = \frac{1}{\beta_0 \ln \frac{Q^2}{\Lambda_{QCD}^2}}\,. \tag{1.1.15}$$

therefore, $\alpha_s(Q^2)$ acquire singularities as long as $Q^2 \leq \Lambda_{QCD}$. The scale Λ_{QCD} is called landau pole. We remark that QCD can be treated as a perturbation theory only if the coupling constant $\alpha_s \ll 1$. This it possible, when the energy scale Q is much larger than Λ_{QCD}. Therefore, the existence of the Landau pole shows that QCD is strongly coupled at low energies, and this is the reason why in Nature isolated quarks or gluons cannot be observed.

1.1.2 Colour confinement

Colour confinement is a key feature of QCD, which states that colour-charged particles cannot be directly observed at energies below approximately 150 MeV. As we have shown in the previous section, the Landau pole, Λ_{QCD} can be taken as an indication of the energy threshold for confinement. Below this scale, only colourless states can be directly detected. These states as known as hadrons and they are bound states of the fundamental particles of the theory, namely quarks and gluons. Hadrons are classified into two categories: baryons, which consist of three quarks (such as the proton), and mesons, which are composed of a quark-antiquark pair. The phenomenon of confinement has been directly observed, but a rigorous proof of QCD confinement is still missing, because at this energy scale, the value of the coupling constant α_s becomes $\gtrsim 1$, spoiling the perturbative techniques.

1.2 Leading Order Factorization

As mentioned in Sec. 1.1.2, quarks and gluons—collectively referred to as partons—cannot be observed as free particles. Therefore, the initial and final states of QCD processes involve hadrons, which are bound states of partons that cannot be described analytically. Consequently, we are only able to measure hadronic cross sections but we can only compute partonic cross sections. To overcome this problem, one can exploit the factorization property of QCD. The goal of this section is to describe this fundamental feature, which allows us to compute hadronic cross sections in terms of partonic cross sections and the experimentally measured Parton Distribution Functions (PDFs). This is just a general overview; several important aspects in the context of Deep Inelastic Scattering (DIS) will be discussed in the following sections, while the factorization of the more complex case of Semi-Inclusive Deep Inelastic Scattering (SIDIS) will be addressed in Sec. 2.1.4.

Factorization is the property of the perturbative QCD that enable us to link experiments with theory. Specifically, it states that at the parton level process can be viewed as a scattering involving a single parton, which carries a longitudinal momentum fraction ξ of the parent hadron.

For instance, we can consider the scattering between a single parton, that carries a momentum $p = \xi P$ and belongs to a hadron with momentum P, and another particle with momentum k, where the parton can be either a quark, an antiquark, or a gluon. In particular, the longitudinal fraction ξ is not fixed, so,through $f_i^{(h)}(\xi)\mathrm{d}\xi$, we can define the parton distribution function (PDF) as the probability to find a parton with a momentum fraction between ξ and $\xi + \mathrm{d}\xi$ inside the hadron h. Therefore, PDFs describe the parton content of the nucleon including all the non-perturbative effects of the process. It can be rigorously proven that the hadronic cross section can be expressed as the sum over all parton species i of the convolution of the partonic cross section for a given initial parton i and the corresponding PDF $f_i^{(h)}$. The property that we have just stated is the so-called factorization, that in formula is expressed as

$$d\sigma^{(h)}(x, Q^2) = \sum_{i=q,\bar{q},g} \int_x^1 \frac{\mathrm{d}\xi}{\xi} f_i^{(h)}(\xi) d\hat{\sigma}_i \left(\frac{x}{\xi}, Q^2 \right) , \tag{1.2.1}$$

where x is the scaling variable of the process (its definition will be provided in the next section), while Q^2 is the energy scale of the process. It is worth mentioning that, throughout the text, partonic quantities are denoted by a hat symbol. Furthermore, we mention that through operator product expansion (OPE) can be rigorously proved the above factorization formula for the DIS process. Thus, we note that PDFs serve as the experimental input in the factorization framework due to their non-perturbative nature. In particular, PDFs describe the internal structure of the incoming hadron, then they are process independent and they can be extracted through fitting procedures potentially from any scattering process.

The factorization result is trivially extended to processes that involve more than one incoming parton, such as hadron collision, than

$$d\sigma^{(h_1,h_2)}(x_1, x_2, Q^2) = \sum_{i,j=q,\bar{q},g} \int_{x_1}^1 \frac{\mathrm{d}\xi_1}{\xi_1} f_i^{(h_1)}(\xi_1) \int_{x_2}^1 \frac{\mathrm{d}\xi_2}{\xi_2} f_j^{(h_2)}(\xi_2) d\hat{\sigma}_{ij} \left(\frac{x_1}{\xi_1}, \frac{x_2}{\xi_2}, Q^2 \right) , \tag{1.2.2}$$

where h_1 and h_2 denote the two incoming hadrons and $\hat{x}_1$ and $\hat{x}_2$ are the scaling variables.

As a final remark, we observe that the above factorization procedure is true only at the leading order (LO) in perturbation theory, as well as the PDFs interpretation as probabilities. At next-to-leading order (NLO), radiative corrections introduce an energy scale dependence into the PDFs, that scale is known as the factorization scale μ_F. The core of the next sections, through the study of DIS process, is to obtain the factorization formula beyond the LO and to investigate the behaviour of the PDFs with respect to the factorization scale μ_F.

1.3 Deep Inelastic Scattering

In this section, we focus on the study of DIS process at the leading order, as it is one of the simplest QCD process, and because it is nothing but the simplest version of the process studied in this thesis: the semi-inclusive deep inelastic scattering. We introduce its kinematics, showing that it depends only on two variables: a hard scale and a dimensionless scaling variable and its hadronic cross section in terms of structure functions. So, we compute the hadronic cross-section at the LO.

1.3.1 DIS kinematic

We now introduce the Deep Inelastic Scattering kinematic, with the meaning of the terms "deep" and "inelastic" clarified in what follows. DIS process is obtained by the scattering of a hadron H and a lepton l, namely

$$l(k) + H(P) \to l(k') + X , \tag{1.3.1}$$

where k and k' are the transferred momenta of the incoming and outgoing leptons respectively, the symbol X denotes a generic final state, that we work with an inclusive cross section, namely, a cross section differential in lepton momentum l' with a sum and integral over all possible states for the X part of the final state. Effectively only the lepton is treated as being detected, we will see that for SIDIS case is slightly different. Furthermore, $q = k - k'$ is the transferred momentum by the virtual gauge boson, which mediates the interactions, then it could be a either a photon or a weak boson. The boson can either be a photon (e.g., if the scattered lepton is an electron) or a weak boson (e.g., if the scattered lepton is a neutrino). A way to visualize the process is shown in Fig. 1.1. In particular, it is showed the case where the interaction is mediated by the electromagnetic force. Specifically, the upper part of the diagram represents the QED interaction at LO between the incoming and outgoing leptons (l and l') and the virtual photon γ^*, while the bubble at the bottom illustrates the contribution from strong interactions. For brevity, for the rest of the chapter, we only investigate the DIS process through the electromagnetic interaction.

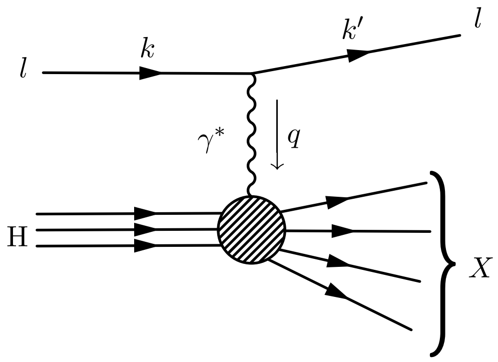


Figure 1.1: DIS Feynman diagram

We now introduce the kinematic variables of the process. Firstly, we define the hard-scale through

$$Q^2 \equiv -q^2 \,, \tag{1.3.2}$$

that is a space-like variable. Then we define the centre of mass energy of the lepton-hadron system

$$\tilde{s} \equiv (P+k)^2 \,. \tag{1.3.3}$$

Hence, from 4-momentum conservation we obtain

$$P + k = P_X + k' \to P_X = P + q \to P_x^2 = m_X^2 = (P+q)^2 \equiv s \,, \tag{1.3.4}$$

where P_x is the 4-momentum of the generic outgoing final state, then m_x its mass, and s is the invariant mass of the hadronic system. At this point, noting that $P_H^2 = M_H^2$,we can clarify the notation "deep" and "inelastic" by specifying the energy regime in which we are working

- **Deep** means $Q^2 \gg M_H^2$, which places us in the ultrarelativistic limit. Hence, we can treat the hadron as massless, as well as its partonic constituents, namely the so-called partons. This frame it is called Parton model picture.

- **Inelastic** means $M_X^2 \gg M_H^2$,implying that the energy is not converted into momentum.

Therefore we can define the further kinematic variable x, the so-called Bjorken-variable, namely

$$x \equiv \frac{Q^2}{2P \cdot q} \,. \tag{1.3.5}$$

As we will see in what follows, the x variable plays a central role in the description of the DIS process.

We now observe that the DIS process has 8 degrees of freedom, given by k' and P_X. However, due to 4-momentum conservation, the fact that k' is massless, and by the rotation symmetry under rotation around the lepton direction we have 6 constraints. Hence, the DIS process is described by $8-5=2$ independent kinematic variables. In conclusion, we can choose to describe the DIS process through the kinematic variables Q^2, x, or equivalently one can define

$$y \equiv \frac{P \cdot q}{P \cdot k} \tag{1.3.6}$$

and describe the process through the variables x, y. Moreover, we note that $Q^2 = xy\tilde{s}$.

As last consideration we observe that

$$M_X^2 = P_X^2 = (P+q)^2 = -Q^2 + 2P \cdot q = Q^2 \frac{1-x}{x} = s \tag{1.3.7}$$

Therefore, $x \in (0,1)$. In particular, we can remark that if $x=1$, then $M_X^2 = 0$, and the process becomes elastic.

In the case of interest in this section, the interaction is mediated by a virtual photon. As a result, the process receives both QED and QCD corrections, and the perturbative expansion is defined with respect to both coupling constants. However, QCD corrections dominate over QED ones, since the strong coupling constant is larger than the QED fine-structure constant. Therefore, we will focus exclusively on QCD corrections, considering only the sub-process $\gamma^* + H \to X$, where γ^* is the virtual photon.

As shown in Fig. 1.1, assuming that the initial lepton is an electron the matrix element DIS process is provided by

$$i\mathcal{M}(eP \to eX) = -(ie)\bar{u}(k')\gamma^\mu u(k) \frac{i}{q^2}(ie) \int \mathrm{d}^4 z\, e^{iq\cdot z} \langle X | j_\mu(z) | P \rangle \,, \tag{1.3.8}$$

where $j^\mu(z)$ is the quark electromagnetic current, $|P\rangle$ is the the initial state of the proton and $\langle X|$ is some high-energy hadronic state. The core of the DIS amplitude is the hadronic matrix element of the current between the proton and a generic high-energy hadronic final state. In particular, we can note that this matrix element contains all the information about the interaction of the electromagnetic current j_μ with the target proton P. In the context of the DIS process, however, we adopt an inclusive approach, meaning that we do not directly measure any specific property of the final state X. Therefore, the hadronic matrix element must be squared and summed over all possible final states. We can thus start from its expression, which is written as

$$i\mathcal{M}(\gamma^* P \to X) = (-ie)\epsilon_\mu(q) \int \mathrm{d}^4 z\, e^{iq\cdot z} \langle X | j_\mu(z) | P \rangle \,. \tag{1.3.9}$$

Firstly, in virtue of the Optical theorem we have

$$\begin{aligned} 2\mathrm{Im}\mathcal{M}(a \to b) &= -i[\mathcal{M}(a \to b) - \mathcal{M}^*(b \to a)] \\ &= \sum_X \int \mathrm{d}\Pi_X \mathcal{M}^*(X \to b)\mathcal{M}(a \to X)\,, \end{aligned} \tag{1.3.10}$$

where $|a\rangle$ and $\langle b|$ are the initial and final states respectively. Therefore, since the Fourier transform of the current is

$$\langle X | j^\mu(q) | P \rangle = \int \mathrm{d}^4 z\, e^{iq\cdot z} \langle X | j^\mu(z) | P \rangle \,, \tag{1.3.11}$$

we obtain

$$
\begin{aligned}
2\mathrm{Im}\mathcal{M}(\gamma^* P \to \gamma^* P) &= \sum_X \int \mathrm{d}\Pi_X |\mathcal{M}(\gamma^* P \to X)|^2 \\
&= \sum_X \int \mathrm{d}\Pi_X \mathcal{M}^*(X \to \gamma^* P)\mathcal{M}(\gamma^* P \to X) \\
&= (ie)^2 \epsilon^*_\mu(q)\epsilon_\nu(q) \left[-\sum_X \int \mathrm{d}\Pi_X \langle P|j^\nu(-q)|X\rangle\langle X|j^\mu(q)|P\rangle \right] . \quad (1.3.12)
\end{aligned}
$$

Hence, we define the tensor

$$
T^{\mu\nu} = \frac{i}{4\pi} \int \mathrm{d}^4 z e^{iq\cdot z} \langle P|T\{j^\mu(z) j^\nu(0)\}|P\rangle \quad (1.3.13)
$$

which is called forward Compton amplitude, since if it is evaluated at $q^2 = 0$ and contracted with the physical polarizations vectors, it gives the forward amplitude for photon-proton scattering

$$
i\mathcal{M}(\gamma^* P \to \gamma^* P) = 4\pi(ie)^2 \epsilon^*_\mu(q)\epsilon_\nu(q)(-iT^{\mu\nu}(P,q)) , \quad (1.3.14)
$$

however, in the present discussion we have to analyze Eq. 1.3.13 for general spacelike q and for general polarization states. Finally, by a direct comparison of Eq. 1.3.12 and Eq. 1.3.14, we obtain

$$
2\mathrm{Im}T^{\mu\nu}(P,q) = \frac{1}{4\pi} \sum_X \int \mathrm{d}\Pi_X \langle P|j^\nu(-q)|X\rangle\langle X|j^\mu(q)|P\rangle . \quad (1.3.15)
$$

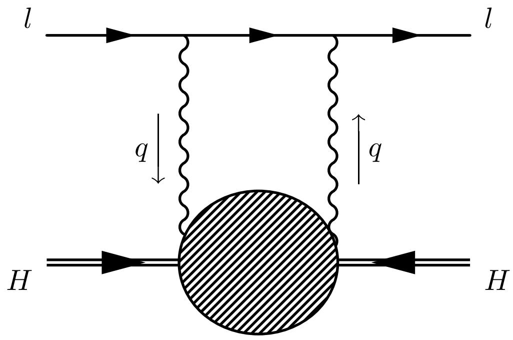


Figure 1.2: DIS squared amplitude

We can now compute the DIS cross section $\sigma(eP \to eX)$ (see Fig. 1.2) in terms of $T^{\mu\nu}$, by averaging over the initial and summing over the final electron spin states, we obtain

$$
\begin{aligned}
\sigma(eP \to eX) =& \frac{4\pi}{2\tilde{s}} \int \frac{\mathrm{d}^3 k'}{(2\pi)^3 2k'^0} \mathrm{e}^4 \frac{1}{2} \sum_{\text{spins}} [\bar{u}(k)\gamma_\mu u(k')\bar{u}(k')\gamma_\nu u(k)] \left(\frac{1}{Q^2}\right)^2 \times \\
& 2\mathrm{Im}T^{\mu\nu}(P,q) . \quad (1.3.16)
\end{aligned}
$$

In general, it is customary to rewrite the squared amplitude as the product of a leptonic tensor $L^{\mu\nu}$ and a hadronic tensor $W^{\mu\nu}$, hence

$$
\sigma(eP \to eX) = \frac{8\pi}{4\tilde{s}} \int \frac{\mathrm{d}^3 k'}{(2\pi)^3 2k'^0} \frac{\mathrm{e}^4}{(Q^2)^2} L_{\mu\nu}(k,k') W^{\mu\nu}(P,q) , \quad (1.3.17)
$$

$$
L^{\mu\nu}(k,k') \equiv \mathrm{Tr}(\not{k}'\gamma^\mu \not{k}\gamma^\nu) = 4(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} k\cdot k') , \quad (1.3.18)
$$

$$
W^{\mu\nu}(P,q) \equiv \mathrm{Im}T^{\mu\nu}(P,q) . \quad (1.3.19)
$$

At this stage, it is useful to convert the integral in Eq. 1.3.16 which depends on the integration variable k'^0 and the scattering angle θ as an integration over the dimensionless variables x and y introduced in the Eqs. 1.3.5 and 1.3.6 respectively. So, it is easy to find that

$$\int \frac{\mathrm{d}^3 k'}{(2\pi)^3 2k'^0} = \int \mathrm{d}x\mathrm{d}y \frac{y\tilde{s}}{(4\pi)^2} \,. \tag{1.3.20}$$

therefore, from Eq. 1.3.16 we find

$$\frac{d^2\sigma}{dxdy}(ep \to eX) = \frac{2\pi\alpha^2 y}{(Q^2)^2} L_{\mu\nu}(k,k') W^{\mu\nu}(P,q) \tag{1.3.21}$$

where $\alpha = \mathrm{e}^2/(4\pi)$ is the electromagnetic fine structure constant. To go further, we have to understand the strcture of the hadronic tensor $W_{\mu\nu}$ (or equivalently of $T_{\mu\nu}$). Therefore, we observe that the cross section must be Lorentz-invariant, implying that the tensor $W_{\mu\nu}$ must also be Lorentz-covariant. Moreover, since the electromagnetic current is conserved, $W_{\mu\nu}$ must satisfy $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$. Finally, as parity is conserved under electromagnetic interactions, the hadronic tensor must be symmetric. Therefore,

$$W_{\mu\nu}(P,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(x,Q^2) + \left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right) W_2(x,Q^2) \,, \tag{1.3.22}$$

where the two scalar functions W_1 and W_2 depend on the two invariants of the problem, x and Q^2. This leads to the equivalent relation for $T^{\mu\nu}$

$$T_{\mu\nu}(P,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) T_1(x,Q^2) + \left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right) T_2(x,Q^2) \,. \tag{1.3.23}$$

We now define the structure functions through the hadronic scalar functions W_1 and W_2

$$F_1(x,Q^2) \equiv W_1(x,Q^2) \tag{1.3.24}$$

$$F_2(x,Q^2) \equiv P\cdot q W_2(x,Q^2) \tag{1.3.25}$$

therefore, the hadronic tensor can be rewritten as

$$W_{\mu\nu}(P,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x,Q^2) + \left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right) \frac{F_2(x,Q^2)}{P\cdot q} \,. \tag{1.3.26}$$

In particular, the structure functions F_i contain all the information about strong interaction. Additionally, if we want to take in account the possibility of process through weak interactions the constraints are weaker, because we lost the parity conservation. Hence, we have a new structure function F_3 through the covariant and antysimmetric term $i\epsilon_{\mu\nu\delta\gamma} q^\delta q^\gamma F_3(x,Q^2)$.

At this point, we are ready to compute the hadronic cross section, than contracting the lepton and the hadronic tensors in Eq. 1.3.21, after some algebra, we obtain

$$\frac{d\sigma}{dxdy} = \frac{4\pi\alpha^2}{Q^2}\left[\frac{\left(1+(1-y)^2\right)}{y} F_1(x,Q^2) + \frac{1-y}{yx}\left(F_2(x,Q^2) - 2xF_1(x,Q^2)\right)\right] \,, \tag{1.3.27}$$

Or equivalently, using $\frac{d}{dy} = x\tilde{s}\frac{d}{dQ^2}$, we have

$$\frac{d\sigma}{dxdy} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1+(1-y)^2\right) F_1(x,Q^2) + \frac{1-y}{x}\left(F_2(x,Q^2) - 2xF_1(x,Q^2)\right)\right] \,, \tag{1.3.28}$$

In conclusion, we note that F_1 corresponds to the absorption of transversely polarized virtual photons, while $F_L = F_2 - 2xF_1$ to the absorption of longitudinally polarized virtual photons.

1.3.2 Leading Order

We now want to apply the factorization theorem reported in Eq.1.2 at the lowest order in perturbation theory, in order to obtain explicit expressions for the structure functions introduced in the previous section. The lowest-order process is simply the scattering off a quark. The interaction between the struck quark and the virtual gauge photon is shown in Fig.1.4, while the overall picture is illustrated in Fig. 1.3.

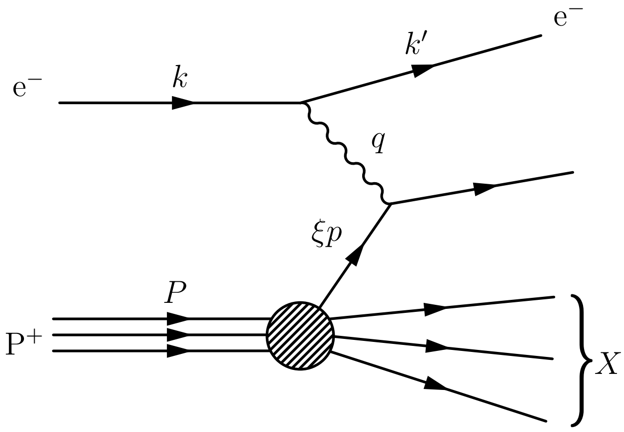


Figure 1.3: DIS Feynman diagram in parton model picture

Firstly, if we define the momentum of the struck parton as $p = \xi P$, then from 4-momentum conservation and since partons are treated as massless, as one can see from Fig. 1.4, we obtain

$$p'^2 = \hat{s} = (p+q)^2 = 2\xi P \cdot q - Q^2 = 0 \to x = \xi \,. \tag{1.3.29}$$

Therefore, at LO the Bjorken variable acquires a clear physical interpretation: it represents the longitudinal momentum fraction of the parent hadron carried by the struck parton. It is now evident that the lower limit of integration in Eq. 1.2.1 corresponds precisely to the Bjorken variable.

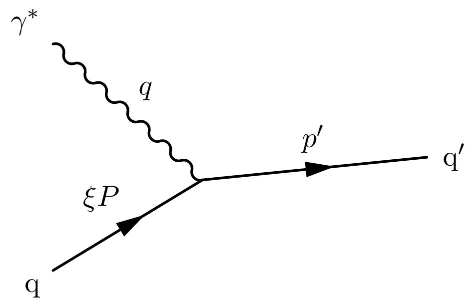


Figure 1.4: DIS Feynman diagram at LO

Therefore, exploiting the factorization property, we need to compute the squared amplitude at the partonic level. We note that At leading order, the process involves a QED vertex, and therefore the interaction between the photon and the quark is purely electromagnetic. It is important to note that, for this reason, contributions from gluon-initiated processes are absent at LO and will only appear at next-to-leading order (NLO), associated with real emissions of quarks. The partonic cross section $\hat{\sigma}(eq \to eq)$ follows from simple QED calculations and is given by:

$$\frac{d\hat{\sigma}}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4} e_q^2 [1 + (1-y)^2] \frac{1}{2} \delta(\xi - x) \,, \tag{1.3.30}$$

from we can read the expressions for the partonic structure functions

$$\widehat{F}_2 = 2x\widehat{F}_1 = x e_q^2 \delta(x - \xi) \tag{1.3.31}$$

Finally, if we compare the cross-section as described in terms of structure function in Eq. 1.3.28 to the cross-section as described through factorization Eq. 1.2.1 using Eq. 1.3.30, we find

$$F_1(x, Q^2) = \sum_{i=q,\bar{q}} \frac{1}{2} e_{q_i}^2 f_i^{(p)}(x) \,, \tag{1.3.32}$$

$$\lim_{Q^2 \to \infty} F_i(x, Q^2) = F_i(x) \quad \text{Bjorken scaling law,} \tag{1.3.33}$$

$$F_2(x) = 2xF_1(x) = \sum_{i=q,\bar{q}} e_{q_i}^2 x f_i^{(p)}(x) \quad \text{Callan-Gross relation.} \tag{1.3.34}$$

In particular, it illustrates how DIS experiments allow us to investigate the internal structure of the proton in terms of its quark and gluon constituents. Secondly, it explicitly shows that the structure functions in the limit of large of large Q^2 are Q^2-independent, this behaviour is the so-called Bjorken scaling law.

It is also worth noting that, in this case, quarks and antiquarks are indistinguishable. Therefore, to independently determine the corresponding PDFs, charged current (CC) processes are necessary.

1.4 Higher-orders Factorization

In this section, we study DIS behaviour at NLO, where IR singularities emerge, thus we provide their physical interpenetration and methods in order to treat them. Thus, we introduce the general factorization formula and the dependence of the PDFs by the μ_F energy scale, which implies the violation of the Bjorken-scaling.

1.4.1 NLO corrections

In the previous section, we computed the leading-order contributions to DIS partonic cross section. We now want to compute the first correction at perturbative order $\mathcal{O}(\alpha_s)$. In fact, at NLO for the DIS process, we need to compute the amplitudes arising from gluon emission. These are represented by the diagrams in Fig.1.7 and Fig.1.11, corresponding to the real and virtual corrections, respectively. However, we have to note that gluons and quarks are treated as massless in the parton model picture, hence integrals appearing in amplitudes calculations and in the phase space manifest infrared (IR) singularities. Two different types of singularities are present, soft and collinear. Whereas, the virtual corrections can be treated using renormalization group arguments, obtaining $Z-$terms corrections by the fields renormalization. Hence, we only need to study the real amplitudes.

We start by computing the real emissions from the incoming quark Fig.1.5. We immediately see that when the intermediate quark becomes nearly on-shell, that is, when the denominator of the propagator approaches zero (i.e., $(p-k)^2 \to 0$), a singularity arises. In particular, since $p^2 = k^2 = 0$ the nature of this singularity can be viewed as follows

$$(p-k)^2 = -2p \cdot k = -2p^0 k^0 (1 - \cos\theta) \qquad , \tag{1.4.1}$$

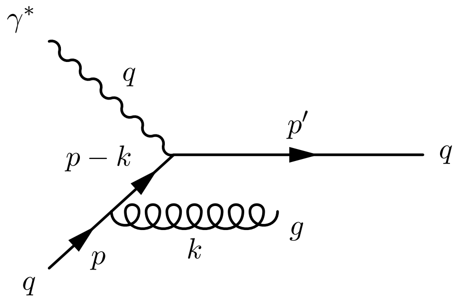


Figure 1.5: Real contribution to the incoming quark [a]

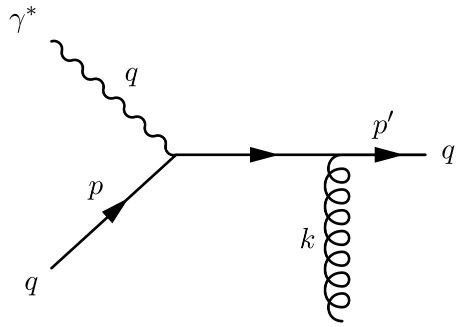


Figure 1.6: Real contribution to the outgoing quark [b]

Figure 1.7: DIS NLO Real gluon corrections

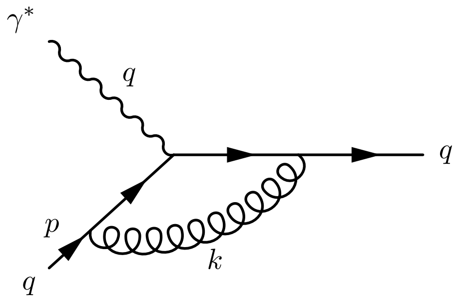


Figure 1.8: Vertex correction

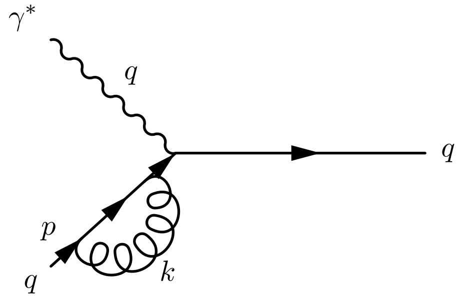


Figure 1.9: Self-energy correction: incoming quark

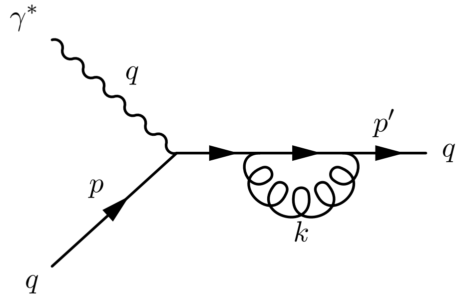


Figure 1.10: Self-energy correction: outgoing quark

Figure 1.11: DIS NLO virtual gluon corrections

than a singularity arise when k is soft either $k^0 \to 0$ or collinear to p, namely $\theta \to 0$ then $k_t^2 \to 0$.

We now focus on the collinear singularity. To this end, we choose the incident quark momentum to lie along the third axis, and the outgoing momenta to lie in the $\hat{1}$–$\hat{3}$ plane. We define z as the fraction of energy carried by the nearly on-shell quark relative to the incoming quark, so that $1-z$ represents the fraction of the initial quark's energy carried away by the gluon. Then the three 4-momenta can be written as

$$p = (p, 0, 0, p) \tag{1.4.2}$$

$$k \approx ((1-z)p, k_t, 0, (1-z)k_t)\,, \tag{1.4.3}$$

$$p - k \approx (zp, -k_t, 0, zp)\,. \tag{1.4.4}$$

These three vectors satisfy $p^2 = k^2 = (p-k)^2 = 0$, up to terms of order k_t^2.

However, in the process involving the emission of a real gluon, both p^2 and k^2 are exactly zero, while $(p-k)^2$ is slightly off-shell, differing from zero by an amount of order k_t^2, then we need to know the value of $(p-k)^2$ in terms of k_t^2 which appears in the propagator. So, we modify the Eq. 1.4.2 in order to satisfy the condition $k^2 = 0$ up to terms of order k_t^4, rewriting k as

$$k = \left((1-z)p, k_t, 0, (1-z)p - \frac{k_t^2}{2(1-z)p} \right) , \tag{1.4.5}$$

then $p - k$ as

$$p - k = \left(zp, k_t, 0, zp + \frac{k_t^2}{2(1-z)p} \right) . \tag{1.4.6}$$

Thus, we obtain

$$(p-k)^2 = -k_t^2 - 2z \frac{k_t^2}{2(1-z)} + \mathcal{O}(k_t^4) . \tag{1.4.7}$$

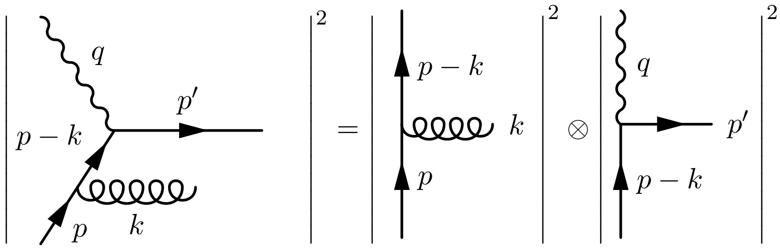
Therefore, if the gluon is real and the quark is virtual, we have

$$k^2 = 0 , \quad (p-k)^2 = -\frac{k_t^2}{1-z} . \tag{1.4.8}$$

Hence, since for $k_t \to 0$ the intermediate quark is almost on-shell we have that,

$$\text{On-shell condition:} \not{p} = \sum_s u^s(p)\bar{u}^s(p) \Longrightarrow \frac{i(\not{p} - \not{k})}{(p-k)^2} = \frac{\sum_s u^s(p-k)\bar{u}^s(p-k)}{(p-k)^2} + \mathcal{O}(k_t^2) , \tag{1.4.9}$$

Therefore, we factorize the collinear singularity in the squared amplitude as follows



$$+ \text{non singular} , \tag{1.4.10}$$

where the first squared amplitude on the r.h.s, is obtained by the matrix element

$$i\mathcal{M}(p \to (p-k), k) = (-ig_s)u(p)T^a\gamma^\mu\epsilon_\mu^*(k)\bar{u}(p-k) , \tag{1.4.11}$$

then exploiting the limit $k_t^2 \to 0$, and using the axial physical gauge, namely

$$\sum_{\lambda=1,2} \epsilon_\mu(k,\lambda)\epsilon_\nu^*(k,\lambda) = \left(-g_{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{n \cdot k} \right) , \tag{1.4.12}$$

one obtains

$$\frac{1}{3}\sum_{\text{colors}} \frac{1}{2}\sum_{\text{pols}}\sum_{\text{spins}} |\mathcal{M}(p \to (p-k), k)|^2 = \frac{2(4\pi)\alpha_s k_t^2 C_F}{z(1-z)} \left[\frac{1+z^2}{1-z} \right] \tag{1.4.13}$$

$$. \tag{1.4.14}$$

At this point, from Eq. 1.4.10, we can express the partonic cross section as follows

$$\hat{\sigma}^{\text{NLO}}(p, q \to k, p') = \frac{1}{(1+v_q)2p2q^0} \int \frac{d^3k}{(2\pi)^3 2k^0} \left| \frac{1}{2} \sum \mathcal{M}(p \to (p-k), k) \right|^2 \times$$

$$\left(\frac{1}{(p-k)^2} \right)^2 \int d\Pi_{p'} \left| \mathcal{M}(q, (p-k) \to p') \right|^2 . \tag{1.4.15}$$

$$\tag{1.4.16}$$

Where v_q is the velocity of q and $\mathrm{d}\Pi_{p'}$ is the phase integral over q'. We now substitute k^0 and $(p-k)$ using Eqs.1.4.2 and 1.4.8, and employ Eq.1.4.2 to rewrite the integral over k as

$$\mathrm{d}^3k = \mathrm{d}k^3\mathrm{d}^2k_t = p\mathrm{d}z\pi\mathrm{d}k_t^2\,. \tag{1.4.17}$$

Then, using Eq. 1.4.13, the cross section can be expressed as

$$\begin{aligned}\hat{\sigma}^{\mathrm{NLO}}(p,q\to k,p') &= \int \frac{p\mathrm{d}z\mathrm{d}k_t^2}{16\pi^2(1-z)p}\left|\frac{1}{2}\sum\mathcal{M}(p\to(p-k),k)\right|^2\frac{(1-z)^2}{k_t^4}\times \\ &\quad \frac{z}{(1+v_q)2zp2q^0}\int \mathrm{d}\Pi_{p'}\left|\mathcal{M}_{p'}\right|^2 \\ &= \frac{\alpha_s}{2\pi}\int \mathrm{d}z\frac{\mathrm{d}k_t^2}{k_t^2}C_F\left[\frac{1+z^2}{1-z}\right]\hat{\sigma}^{(0)}(zp)\end{aligned} \tag{1.4.18}$$

thus, we define the so-called splitting function as

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{1-z}\right]\,, \tag{1.4.19}$$

then

$$\hat{\sigma}^{\mathrm{NLO}} = \frac{\alpha_s}{2\pi}\int_0^{k^2_{t_{max}}}\frac{\mathrm{d}k_t^2}{k_t^2}\int \mathrm{d}zP_{qq}(z)\hat{\sigma}^{(0)}(zp)\,. \tag{1.4.20}$$

Furthermore, we remember that the incoming quark carries a fraction ξ of the momentum proton P, then, since p' is massless, by momentum conservation we note that the struck quark and the photon must satisfies

$$(\xi zP+q)^2 = 0 \to z = \hat{x} = \frac{x}{\xi} \tag{1.4.21}$$

where $\hat{x}$ is the corrective of the Bjorken variable at the parton level. Thus

$$\hat{\sigma}^{\mathrm{NLO}} = \frac{\alpha_s}{2\pi}\int_0^{k^2_{t_{max}}}\frac{\mathrm{d}k_t^2}{k_t^2}\int \mathrm{d}\hat{x}P_{qq}(\hat{x})\hat{\sigma}^{(0)}(\hat{x}p)\,. \tag{1.4.22}$$

We now have reached a crucial point, because we recognize the logarithmic collinear singularity, in the integral $\int_0^{k^2_{t_{max}}}\frac{\mathrm{d}k_t^2}{k_t^2}$. Therefore, to prevent the appearance of divergences in the cross section we are required to introduce an IR cut-off Λ. Consequently, collinear singularities are managed similarly to UV divergences through renormalization, being absorbed into the parton distribution functions. This absorption leads to the emergence of a scale dependence in the PDFs.

However, before proceeding further, we observe that a divergence also arises in the limit $\hat{x}\to 1$, corresponding to the case where the emitted gluon becomes soft. Nevertheless, we have not yet taken into account the other contributions to the amplitude, in particular the virtual corrections. Specifically, This correction contributes only when $\hat{x} = 1$. At first, we note that since all of the virtual diagrams contain the factor $\delta((\xi P+q)^2)$, their contribution to the structure function is proportional to $\delta(1-\hat{x})$, hence P_{qq} is modified as follows

$$P_{qq} = C_F\left[\frac{1+\hat{x}^2}{1-\hat{x}}\right] + K\delta(1-\hat{x})\,, \tag{1.4.23}$$

with k a constant to be determined. To this aim, we require that $P(\hat{x})$ cannot vary with Q^2, so its integral in $\hat{x}$ must be zero. Therefore, the splitting function becomes

$$P_{qq}(\hat{x}) = C_F\left(\frac{1+\hat{x}^2}{(1-\hat{x})_+} + \frac{3}{2}\delta(1-\hat{x})\right)\,, \tag{1.4.24}$$

where "plus" denotes the plus distribution defined as in the appendix A.2.1.

Finally, concerning the real corrections we note that so far, we have only considered the squared amplitude from the squaring of the diagram of Fig. 1.5. In fact, in the light-cone gauge the $|a|^2$ terms is the only which gives a logarithmic divergences – the other terms, ab^*, b^*a, $|b|^2$, as can be seen by using dimensional analysis, give finite corrections to the structure function; all the possible contributions to the squared amplitude are reported in Fig. 1.15.

In conclusion, the soft divergence, i.e. $\hat{x} \to 1$, cancels out in the sum of the virtual and real contributions. While, the UV-divergences are treated by using renormalization techniques. However, the collinear divergence remains, hence we are left only with IR-divergences. Theorems [10] and [11] ensure the cancellation of both soft and collinear singularities between the real and virtual gluon diagrams. In particular, they state that suitably defined inclusive quantities will be free of singularities in the massless limit, which is the case for the gluon radiation emitted by the outgoing quark. However, when considering the radiation emitted by the incoming quark, we are not inclusive. Indeed, the photon scatters off a quark with a definite momentum, and so it is sensitive to collinear splitting. In other words, The virtual photon can differentiate between a single quark and a collinear quark-gluon pair that share the same total momentum. From a physical perspective, the limit $k_t \to 0$ corresponds to the long-range ("soft") component of the strong interaction, which cannot be computed within perturbation theory.

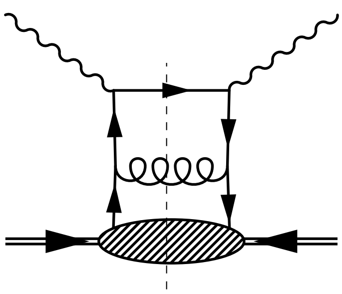
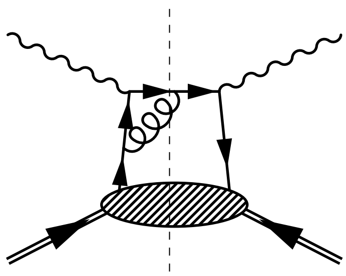
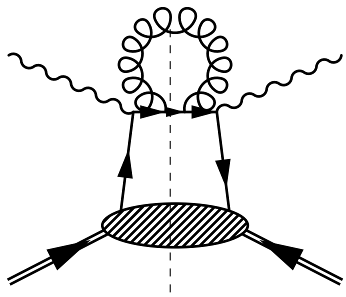


Figure 1.12: Direct contribution $|a|^2$



+ complex conjugate

Figure 1.13: Interference contribution ab^*



Including virtual corrections $\sim$ field renormalization for final quark

Figure 1.14: Direct contribution $|b|^2$

Figure 1.15: DIS gluon corrections

1.4.2 Factorization of collinear singularities

The crucial idea behind the removal of collinear singularities lies in recognizing that the small-k_t limit reflects a sensitivity to long-distance strong interactions, which cannot be addressed within perturbative QCD. The parton distribution functions originally introduced in the Sec. 1.2 are unphysical, bare quantities, and can be redefined to absorb the divergent contributions, yielding finite, physical PDFs. Thus, the factorization formula introduced in Eq. 1.2.1 needs a modification.

Specifically, the DIS differential partonic cross-section through Eq. 1.4.22 can be rewritten as follows

$$\frac{d\hat{\sigma}_q^{(1)}}{dxdQ^2} = e_q^2 \frac{\alpha_s}{2\pi} \left[P_{qq}(\hat{x}) \ln \frac{Q^2}{\Lambda^2} + C(\hat{x}) \right] , \tag{1.4.25}$$

where Λ is the scale introduced in order to regularize the logarithmic divergence, $C(\hat{x})$ is the non-logarithmic contribution to the cross section (sees Eq. 1.4.10) and it is a calculable function, and P_{qq} is the splitting function defined as in Eq. 1.4.24.

By considering the structure function F_2 and, for simplicity, focusing on a single parton species, we can express the result up to $\mathcal{O}(\alpha_s)$. Thus, by adding the leading-order contribution from Eq.1.3.32 and performing the change of variables $\hat{x} = x/\xi$ in Eq.1.2.1, we obtain:

$$F_2(x,Q^2) = xe_q^2\left[f_q^{(p),(0)}(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}}\left(P_{qq}(\hat{x})\ln\frac{Q^2}{\Lambda^2} + C(\hat{x})\right) f_q^{(p),(0)}\left(\frac{x}{\hat{x}}\right) + \mathcal{O}(\alpha_s^2)\right], \quad (1.4.26)$$

where the subscript (0) means bare. Then, in completely analogy to the renormalization procedure, we introduce a factorization scale μ_F in order to split the divergent contribution

$$\ln\frac{Q^2}{\Lambda^2} = \ln\frac{Q^2}{\mu_F^2} + \ln\frac{\mu_F^2}{\Lambda^2}\,. \quad (1.4.27)$$

Thus, Eq. 1.4.26 can be rewritten as

$$\begin{aligned}F_2(x,Q^2) = xe_q^2\Bigg[f_q^{(p),(0)}(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}}\Bigg(P_{qq}(\hat{x})\ln\frac{\mu_F^2}{\Lambda^2} + P_{qq}(\hat{x})\ln\frac{Q^2}{\mu_F^2}\\ +C(\hat{x})\Bigg) f_q^{(p),(0)}\left(\frac{x}{\hat{x}}\right) + \mathcal{O}(\alpha_s^2)\Bigg]\,. \quad (1.4.28)\end{aligned}$$

The physical PDF is obtained introducing the μ_F-dependent term δf

$$\delta f\left(x, \frac{Q^2}{\Lambda^2}\right) \equiv \frac{\alpha_s}{2\pi}\int_x^1 \mathrm{d}f_q^{(p),(0)}\left(\frac{x}{\hat{x}}\right) P_{qq}(\hat{x})\ln\frac{\mu_{\mathrm{F}}^2}{\Lambda^2} \quad (1.4.29)$$

thus the physical PDF have a μ_F dependence and it is

$$f^{(p)}(x,Q^2) = f^{(p),(0)}(x) + \delta f\left(x, \frac{Q^2}{\Lambda^2}\right) + \mathcal{O}(\alpha_s^2)\,. \quad (1.4.30)$$

Thus, restoring the sum over the possible flavours we obtain

$$F_2(x,Q^2) = \sum_i e_{q_i}^2 \int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}} x f_i^{(p)}\left(\frac{x}{\hat{x}}, \mu_{\mathrm{F}}\right)\left(\delta(1-\hat{x}) + \frac{\alpha_s}{2\pi}\left[P_{qq}(\hat{x})\ln\frac{Q^2}{\mu_{\mathrm{F}}^2} + R(\hat{x})\right]\right) \quad (1.4.31)$$

$$\underset{\substack{\text{generalize}\\ \text{to all orders}\\ \text{in expansion in } \alpha_s}}{\Longrightarrow} \quad \sum_i e_{q_i}^2 \int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}} x f_i^{(p)}\left(\frac{x}{\hat{x}}, \mu_{\mathrm{F}}\right) C_i\left(\hat{x}, \alpha_s, \ln\frac{\mu_{\mathrm{F}}^2}{Q^2}\right)\,. \quad (1.4.32)$$

The above formula is a prototype of the so-called factorization formula. The sum runs over $i = q$, where q represents all possible quark flavours. The PDFs f_i contain long-distance effects, including non-perturbative corrections of order $\mathcal{O}\left(\frac{\Lambda_{QCD}}{Q^2}\right)$. Since these contributions cannot be computed through perturbative calculations, they must be extracted from data. In contrast, the functions C_i, known as coefficient functions, encode all short-distance effects and can be computed as a series expansion in $\alpha_s(Q) \ll 1$. In other words, they are derived from Feynman amplitudes at the partonic level. In particular, we note that since the dependence by Q of the PDFs, one the important consequences of the factorization procedure is the violation of the Bjorken scaling beyond the leading order calculations.

However, we note that the above formula is not complete yet. Indeed, we take in account the case where we have an incoming gluon, so a PDF of the type $f_g(\xi)$. Hence, we could have another possible partonic reaction $\gamma^* g \to q\bar{q}$. The calculation of this contribution at the NLO proceed in the same way as the $\gamma^* q \to gq$ calculation, and we obtain the complete factorization formula:

$$F_2(x,Q^2) = x \sum_{q,\bar{q}} e_{q_i}^2 \int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}} f_q^{(p)}\left(\frac{x}{\hat{x}}, \mu_{\mathrm{F}}\right) \left(\delta(1-\hat{x}) + \frac{\alpha_s(\mu_{\mathrm{R}})}{2\pi} C_q^{(1)}\left(\hat{x}, \ln\frac{\mu_{\mathrm{F}}^2}{Q^2}, \ln\frac{\mu_{\mathrm{R}}^2}{Q^2}\right) + \dots\right)$$

$$x \sum_{q,\bar{q}} e_{q_i}^2 \int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}} f_g^{(p)}\left(\frac{x}{\hat{x}}, \mu_{\mathrm{F}}\right) \left(\frac{\alpha_s(\mu_{\mathrm{R}})}{2\pi} C_g^{(1)}\left(\hat{x}, \ln\frac{\mu_{\mathrm{F}}^2}{Q^2}, \ln\frac{\mu_{\mathrm{R}}^2}{Q^2}\right) + \dots\right). \quad (1.4.33)$$

We note that we also include the dependence on the renormalization scale, on which the coefficient functions depend due to loop contributions to the partonic amplitude. However, since the arbitrariness of the two scale we are always free to impose $\mu = \mu_{\mathrm{F}} = \mu_{\mathrm{R}}$. The analytical expressions for the the coefficient functions in the $\overline{\mathrm{MS}}$ renormalization scheme at $\mathcal{O}(\alpha_s)$ were first computed in [12]. Finally, we can repeat the process for the others structure functions present in the formulation of the cross section (e.g. F_1,F_3...).

In conclusion, since the factorization theorem explained above, we can express the differential hadronic cross section as follows

$$\frac{d\sigma_{l+h\to l+X}}{dxdQ^2}(x,Q^2) = \sum_{i=q,\bar{q},g} \int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}} f_i^{(h)}\left(\frac{x}{\hat{x}}, \mu_{\mathrm{F}}\right) \hat{\sigma}_{l+i\to l+X}\left(\hat{x}, \alpha_s(\mu_{\mathrm{R}}), \ln\frac{\mu_{\mathrm{F}}^2}{Q^2}\right). \quad (1.4.34)$$

Here, σ is the physical cross section, namely the hadronic one; therefore, it must not depend on the factorization scale. On the other hand, $\hat{\sigma}$ is the partonic cross section with collinear corrections removed, so it can be computed from the partonic Feynman diagrams of the process. Meanwhile, $f_i^{(h)}$ represents the PDF for the parton i with respect to the hadron h. Lastly, to ensure perturbative stability, one should set $\mu \sim Q$. One then needs to determine how $f_i(\xi, \mu = Q)$ evolves as Q varies, this goal will be achieved in Sec. 1.5.

1.4.3 Dimensional regularization and soft logarithms

Before to go further with the study of the PDF dependence by the factorization scale, we briefly review the technique used to regularize higher order corrections beyond NLO, so multiple emissions.

At higher orders in the perturbative expansion, it is not strictly necessary to explicitly split the integral over k_t^2 into two separate regions. Instead, using dimensional regularization is more convenient, as it preserves the Lorentz and gauge invariance of the theory, unlike the cut-off method. Moreover, this approach highlight the structure of the singularities that appear in the calculations. In this section, we provide its definition and show how, by means of a fundamental distributional identity, it leads to plus distributions and logarithms.

In the partonic cross-section computation, we have two different types of integrals: loop integrals and phase-space integrals, which are performed in four dimensions. The core idea of dimensional regularization is to compute these integrals in $d = 4 - 2\epsilon$ dimensions, with the limit $\epsilon \to 0$ to be taken at the end of the calculation. In the end, the infrared singularities manifest as simple poles in ϵ. We can now distinguish two cases:

- $\epsilon > 0$ in order to extract the UV divergences;
- $\epsilon < 0$ in order to extract the IR divergences.

In this section we are interested in the second case.

The phase space in d-dimension for a DIS process with $n-$outgoing partons it is expressed as follows

$$\mathrm{d}\phi_n(\epsilon) = \prod_{i=1}^{n} \frac{\mathrm{d}^{d-1}p_i}{(2\pi)^{d-1}(2p_i^0)}\delta^{(d)}\left(p+q-\sum_j p_j\right) \tag{1.4.35}$$

Therefore, the divergent integrals becomes finite but they acquire a term proportional to $1/\epsilon$. ϵ-finite terms come from the interference of poles with exponentials in ϵ. For instance, we can take the regularized collinear-divergent integral

$$\mathcal{I}_{\text{coll}} = \int_0^{Q^2} \frac{\mathrm{d}^2 k_t}{k_t^2} \to \tilde{\mu}^{2\epsilon}\int_0^{Q^2}\frac{\mathrm{d}k_t^2}{(k_t^2)^{1+\epsilon}}\,, \tag{1.4.36}$$

where $\tilde{\mu}$ is usually defined as the factorization scale multiplied by some constant; indeed, in order to keep the mass dimension of the integral the same as in $d = 4$, we multiply it by $\tilde{\mu}^{4-d}$, where $\tilde{\mu}$ is parameter of mass dimension one and arbitrary value. Therefore,

$$\mathcal{I}_{\text{coll}} = \tilde{\mu}^{2\epsilon}\int_0^{Q^2}\frac{\mathrm{d}k_t^2}{(k_t^2)^{1+\epsilon}} = -\frac{1}{\epsilon}\left(\frac{\tilde{\mu}^2}{Q^2}\right)^{\epsilon}\,, \tag{1.4.37}$$

and by a simple Taylor expansion we obtain

$$\mathcal{I}_{\text{coll}} = \frac{1}{\epsilon}\left(-1+\epsilon\ln\frac{Q^2}{\tilde{\mu}^2}+\mathcal{O}(\epsilon^2)\right) = -\frac{1}{\epsilon}+\ln\frac{Q^2}{\tilde{\mu}^2}+\mathcal{O}(\epsilon)\,, \tag{1.4.38}$$

On the other hand, in this context in the calculation of the real gluon emission contributions the term $(1-\hat{x})^{-1-\epsilon}$ appears at all order because of the collinear and soft emission [13]. In particular, the plus distribution arise from this term, in fact (for the proof see the appendix A.2.1)

$$(1-\hat{x})^{-1-\epsilon} = \frac{\delta(1-\hat{x})}{\epsilon} + \sum_{i=0}^{\infty}\frac{1}{i!}\epsilon^i\left[\frac{\ln^i(1-\hat{x})}{1-\hat{x}}\right]_+\,, \tag{1.4.39}$$

depending on the order of the pole multiplying the identity, a different finite logarithm survives, while the remaining terms are either cancelled or vanish as $\epsilon \to 0$. Thus, the IR-soft poles cancel, while the collinear poles are cancelled against the poles present in the PDFs. In the $\overline{\text{MS}}$ scheme, $f_i^{(h)}(\xi,\mu)$ is defined via minimal subtraction of the $1/\epsilon$ pole. On the other hand, at the $n-$perturbative order C_i and $\hat{\sigma}$ contain the terms

- $\alpha_s(\mu)^n \ln^n \frac{Q^2}{\mu^2}$ and when are $\mathcal{O}(1)$ they spoil the perturbative approach. However, it can be dealt by evolving PDFs at the hard scale of the process as it shown in the next section.
- $\alpha_s(\mu)^n\left[\frac{\ln^k(1-\hat{x})}{1-\hat{x}}\right]_+$ where usually $0 \le k \le 2n-1$.

In this case, we recognize that these plus-distributions regularize the soft divergence at $\hat{x} = 1$. Specifically, they represent what remains in the calculation after the cancellation of both the soft and collinear divergences. There is, however, an important aspect to consider: IR singularities cancel out completely, but near the singular point (though not exactly at it), a large logarithm remains due to the cancellation of the singularity itself. Roughly speaking when $\hat{x}$ is equal or larger than a value $\bar{x}$ satisfying

$$\alpha_s \ln^2(1-\bar{x}) \sim \mathcal{O}(1) \tag{1.4.40}$$

any finite order truncation would be meaningless, since all terms in the perturbative series are of the same order. Since $\hat{x} \in (0,1)$, this kinematic region where $\hat{x} \to 1$ is always contained within the definition of the cross-section, so that the coefficient functions C_i need each time to be resummed.

Clearly, this implies that soft-gluon effects may persist and become significant in kinematic regions where there is a large imbalance between real and virtual contributions. In such cases, a fixed-order expansion becomes unreliable, as higher-order terms can have a non-negligible weight. Therefore, to obtain reliable predictions, calculations to all orders in perturbation theory are required. One can note that these singularities—and the logarithms associated with them—can be computed to all orders. This is where the threshold resummation theory comes into play, as it allows for the summation of these logarithms. We obtain the resummation formula trough renormalization group approach in sec. 3 and we apply it to the SIDIS process, which is the main process studied throughout this thesis.

1.5 DGLAP equation

The dependence of the PDFs is through two variables, $\hat{x}$ and μ_{F}. The first one is non-perturbative, but the dependence on the factorization scale can be computed. Indeed, the dependence by μ_F is through the collinear logarithmic terms, as we have shown in Eq. 1.4.30. For this purpose, we can use a RGE approach, as done in Sec. 1.1.1 for the strong coupling. In this case, the renormalization group evolution is known as the Altarelli-Parisi (or DGLAP) evolution. To derive this equation, we use the fact that the hadronic cross section is a physical observable, which implies that it cannot depend on μ_{F}. So, using Eq. 1.4.34, we obtain:

$$\mu_{\mathrm{F}}^2 \frac{d}{d\mu_{\mathrm{F}}^2}\left(\frac{d\sigma}{dxdQ^2}\right) = \sum_{i=q,\bar{q},g} \int_x^1 \mathrm{d}\hat{x} \left(\mu_{\mathrm{F}}^2 \frac{\partial f_i^{(h)}}{\partial \mu_{\mathrm{F}}^2}\hat{\sigma} + f_i^{(h)} \mu_{\mathrm{F}}^2 \frac{\partial \hat{\sigma}}{\partial \mu_{\mathrm{F}}^2}\right) = 0\,, \tag{1.5.1}$$

where, for simplicity, we omitted the dependences. Since the r.h.s vanishes and $\hat{\sigma}$ can be obtained by perturbative calculations we can extract $\frac{\mathrm{d}f_i^{(h)}}{\mathrm{d}\log\mu_{\mathrm{F}}^2}$. For example at LO in the strong coupling, i.e. $\mathcal{O}(\alpha_s)$, from Eq.1.4.31, we have

$$0 = \int_x^1 \mathrm{d}\hat{x} \frac{x}{\hat{x}} \mu_{\mathrm{F}}^2 \frac{\partial}{\partial \mu_{\mathrm{F}}^2}\left(f_i^{(p)}\left(\frac{x}{\hat{x}},\mu_{\mathrm{F}}\right)\delta(1-\hat{x})\right) + \frac{x}{\hat{x}} f_i^{(p)}\left(\frac{x}{\hat{x}},\mu_{\mathrm{F}}\right)\frac{\alpha_s(\mu_{\mathrm{R}})}{2\pi}\left(-P_{qq}(\hat{x})\right) + \mathcal{O}(\alpha_s^2)\,, \tag{1.5.2}$$

and dividing by x and defining $y \equiv x/\hat{x} \to \mathrm{d}\hat{x}/\hat{x} = -\mathrm{d}y/y$, we obtain

$$\mu_{\mathrm{F}}^2 \frac{\partial}{\partial \mu_{\mathrm{F}}^2} f_i^{(p)}(x,\mu_{\mathrm{F}}) = \frac{\alpha_s(\mu_{\mathrm{R}})}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y} P_{qq}^{(0)}\left(\frac{x}{y}\right) f_i^{(p)}(y,\mu_{\mathrm{F}})\,, \tag{1.5.3}$$

hence, the DGLAP equation is an integro-differential equation. Here, we obtain the simplest case, namely, up to $\mathcal{O}(\alpha_s)$, while the role of the subscript (0) will be clarified in the next steps. It can be shown [14] that the general formula, including $\mathcal{O}(\alpha_s^n)$ corrections and all partons, takes the form of a $(2N_f+1)$-dimensional matrix equation acting on the space of quarks, antiquarks, and gluons, namely

$$\mu_{\mathrm{F}}^2 \frac{\partial}{\partial \mu_{\mathrm{F}}^2}\begin{pmatrix} f_{q_i}^{(h)}(x,\mu_{\mathrm{F}}) \\ f_g^{(h)}(x,\mu_{\mathrm{F}})\end{pmatrix} = \frac{\alpha_s(\mu_{\mathrm{R}})}{2\pi}\sum_{q_j,\bar{q}_j}\int_x^1 \frac{\mathrm{d}y}{y}\begin{bmatrix} P_{q_iq_j}\left(\frac{x}{y},\alpha_s(\mu_{\mathrm{R}})\right) & P_{q_ig}\left(\frac{x}{y},\alpha_s(\mu_{\mathrm{R}})\right) \\ P_{gq_j}\left(\frac{x}{y},\alpha_s(\mu_{\mathrm{R}})\right) & P_{gg}\left(\frac{x}{y},\alpha_s(\mu_{\mathrm{R}})\right)\end{bmatrix}\begin{pmatrix} f_{q_i}^{(h)}(y,\mu_{\mathrm{F}}) \\ f_g^{(h)}(y,\mu_{\mathrm{F}})\end{pmatrix}\,, \tag{1.5.4}$$

which can be written in a more compact form as

$$\mu_{\mathrm{F}}^2 \frac{\partial f_i^{(h)}(x,\mu_{\mathrm{F}})}{\partial \mu_{\mathrm{F}}^2} = \frac{\alpha_s(\mu_{\mathrm{R}})}{2\pi} \sum_j P_{ij} \otimes f_j^{(h)}(x,\mu_{\mathrm{F}}) \,, \tag{1.5.5}$$

where the sum is over all the active flavours and g, while i could be either a flavour index or a gluon index g.

We observe that each splitting function (or evolution kernel) is also computable as a series expansion in α_s.

$$\begin{aligned}
P_{q_i q_j}(z,\alpha_s) &= \delta_{ij} P_{qq}^{(0)}(z) + \frac{\alpha_s}{\pi} P_{q_i q_j}^{(1)}(z) + \dots \\
P_{qg}(z,\alpha_s) &= P_{qg}^{(0)}(z) + \frac{\alpha_s}{\pi} P_{qg}^{(1)}(z) + \dots \\
P_{gq}(z,\alpha_s) &= P_{gq}^{(0)}(z) + \frac{\alpha_s}{\pi} P_{gq}^{(1)}(z) + \dots \\
P_{gg}(z,\alpha_s) &= P_{gg}^{(0)}(z) + \frac{\alpha_s}{\pi} P_{gg}^{(1)}(z) + \dots
\end{aligned} \tag{1.5.6}$$

In order to obtain the splitting functions at higher orders, we must include higher-order corrections in α_s by adding multiple gluon emissions to the diagrams in Tab. 1.2. For instance, the splitting function P_{qq} at higher orders is derived from the first diagram in Tab. 1.2 inserting on it multiple gluon emissions on the quark leg. Specifically, the amplitude for $P_{qq}^{(1)}$ is given by the diagram with two real gluon emissions plus the diagrams with one gluon emission accompanied by a gluon loop correction. The same argument also holds for the other splitting function. Whereas, at the leading order we have the results reported in Tab. 1.2.

As a final remark, we emphasize that, due to the $SU(N_f)$ flavour symmetry and charge conjugation invariance, we have

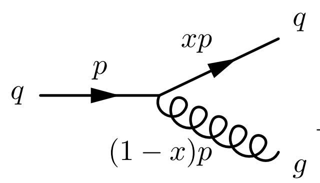
$$\begin{aligned}
P_{q_i q_j} &= P_{\bar{q}_i \bar{q}_j} \\
P_{q_i \bar{q}_j} &= P_{\bar{q}_i q_j} \\
P_{q_i g} &= P_{\bar{q}_i g} \equiv P_{qg} \\
P_{g q_i} &= P_{g \bar{q}_i} \equiv P_{gq} \,,
\end{aligned} \tag{1.5.7}$$

Specifically, the splitting functions P_{qg} and P_{gq} are flavour-independent and the same for both quarks and antiquarks. As we shown in the previous steps, the leading-order $P_{q_i q_j}$ splitting function is zero unless $q_i = q_j$.

1.5.1 Splitting functions

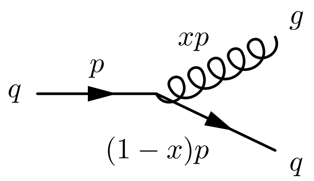
In this section, we provide a physical interpretation of the splitting functions and analyse their structure beyond LO.

The leading-order splitting functions $P_{ab}^{(0)}$ have a well-defined physical interpretation: they describe the probability of finding a parton of type a inside a parton of type b, carrying a fraction x of the parent parton's longitudinal momentum, while having a transverse momentum squared much smaller than μ^2. Thus, their interpretation as probabilities requires them to be positive

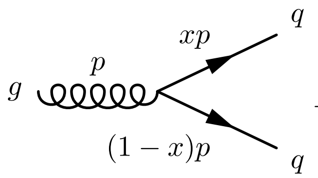


$$\to P^{(0)}_{qq} = C_F \left(\frac{1+x^2}{[1-x]_+} + \frac{3}{2}\delta(1-x) \right)$$

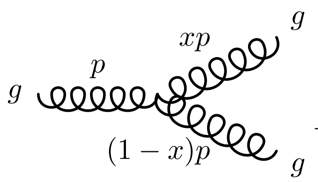
+ virtual contribution



$$\to P^{(0)}_{gq} = C_F \left(\frac{1+(1-x)^2}{x} \right) = P^{(0)}_{qq}(x \to 1-x)$$



$$\to P^{(0)}_{qg} = T_R \left(x^2 + (1-x)^2 \right) \quad T_R = \frac{1}{2}$$



$$\to P^{(0)}_{gg} = 2C_A \left(\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) + \delta(1-x)\frac{11C_A - 4N_f T_R}{6} \right)$$

+ virtual contribution

Table 1.2: Splitting functions at leading order

definite for $x \leq 1$ and to satisfy the following sum rules:

$$\int_0^1 \mathrm{d}x P^{(0)}_{qq}(x) = 0\,, \tag{1.5.8}$$

$$\int_0^1 \mathrm{d}x \left[P^{(0)}_{qq}(x) + P^{(0)}_{gq}(x) \right] = 0\,, \tag{1.5.9}$$

$$\int_0^1 \mathrm{d}x \left[2N_f P^{(0)}_{qg}(x) + P^{(0)}_{gg}(x) \right] = 0\,, \tag{1.5.10}$$

this corresponds to quark number conservation and momentum conservation in quark and gluon splittings, respectively.

Beyond the LO, the flavour structure of $P_{q_i q_j}$ is no longer trivial. Through the $SU(N_f)$ flavour symmetry, we can rewrite the splitting functions in terms of flavour singlet (S) and non-singlet (V) contributions

$$P_{q_i q_j} = \delta_{ij} P^V_{qq} + P^S_{qq}\,, \tag{1.5.11}$$

$$P_{q_i \bar{q}_j} = \delta_{ij} P^V_{q\bar{q}} + P^S_{q\bar{q}}\,, \tag{1.5.12}$$

At NLO, the pure singlet functions are non-zero, but on the other hand, we have the symmetry condition:

$$P^S_{qq} = P^S_{q\bar{q}} \tag{1.5.13}$$

The formal solution of the GLAP equation requires a treatment beyond the scope of this work. However, we note that, in its current form, the equations for individual PDFs cannot be decoupled. Therefore, as previously mentioned, it is necessary to introduce appropriate combinations of PDFs, which are classified into two categories: non-singlet PDFs and singlet PDFs. Consequently, since the coefficient functions are convoluted with the PDFs, they are also classified accordingly.

As we have shown, the DGLAP equations (Eq. 1.5.5) form a system of $2N_f + 1$ coupled equations. Therefore, using the notation $f_{q_i}^{(h)} = q_i$ and $f_g^{(h)} = g$, it is convenient to rewrite the quark sector in terms of the flavour singlet (S) and non-singlet (NS) components to decouple the DGLAP equations. Hence, we define the singlet term as

$$q_S \equiv \frac{1}{N_f} \sum_i^{N_f} (q_i + \bar{q}_i)\,, \tag{1.5.14}$$

which evolves together with g according to:

$$\frac{\partial}{\partial Q^2} \begin{pmatrix} q_S \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_S \\ g \end{pmatrix}\,, \tag{1.5.15}$$

while the NS sector is given by three non-singlet combinations of PDFs, namely

$$q^{\pm}_{\mathrm{NS},ik} \equiv q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k) \tag{1.5.16}$$

$$q^{V}_{\mathrm{NS}} \equiv q_i - \bar{q}_i \tag{1.5.17}$$

which evolve independently with P^{+}_{NS}, P^{-}_{NS}, P^{V}_{NS} and decouple the remaining $2N_f - 1$ equations. Specifically, we have

$$P^{\pm}_{\mathrm{NS}} = P^{V}_{qq} \pm P^{V}_{q\bar{q}} \tag{1.5.18}$$

$$P^{V}_{\mathrm{NS}} = P^{V}_{qq} - P^{V}_{q\bar{q}} + N_f(P^{S}_{qq} - P^{V}_{q\bar{q}}) \equiv P^{-}_{\mathrm{NS}} + P^{S}_{\mathrm{NS}} \tag{1.5.19}$$

all the splitting functions are known up to NNLO [15–17].

For instance, we can analyze the DIS factorization formula using these redefined PDFs. From Eq. 1.4.34, we know that the DIS structure functions, in the case of the electromagnetic interaction, are given by the convolution

$$F_k^{\mathrm{DIS}} = \sum_j \left(C_k^{q_j} \otimes q_j + C_k^{\bar{q}_j} \otimes \bar{q}_j \right) + C_k^{g} \otimes g \tag{1.5.20}$$

$$= \sum_j e^2_{q_j} C_k^{\mathrm{NS}} \otimes q_j^{\mathrm{NS}}(x, Q^2) + \left(\sum_j e^2_{q_j} \right) \left[C_k^{S} \otimes q_S + C_k^{g} \otimes g \right] (x, Q^2) \tag{1.5.21}$$

where $k = 1, L$, the sums run the active flavours, e_{q_j} are the electromagnetic charges of quarks. The flavour combination q_j^{NS} is defined as follows

$$q_j^{\mathrm{NS}} \equiv \frac{1}{N_f} \sum_{k=1}^{N_f} q^{+}_{\mathrm{NS},jk} = (q_j + \bar{q}_j) - \frac{1}{N_f} \sum_{k=1}^{N_f} (q_k + \bar{q}_k) \tag{1.5.22}$$

It evolves with P^{+}_{NS}, while q_s, defined in Eq. 1.5.14, follows the evolution dictated by Eq. 1.5.5. The equivalence between Eqs. 1.5.20 and 1.5.21 directly arises from the charge conjugation symmetry $C_k^{q_i} = C_k^{\bar{q}_i}$ in the context of the electromagnetic interaction. Consequently, we can separate NS diagrammatic contributions from pure-singlet (PS) ones and express the result as follows:

$$C_k^{q_i} = C_k^{\bar{q}_i} = e_{q_i}^2 C_k^{\mathrm{NS}} + \frac{1}{N_f}\left(\sum_j e_{q_j}^2\right) C_k^{\mathrm{PS}} \tag{1.5.23}$$

In this case, NS contributions are defined as those in which, on either the left or right side of the cut diagrams, the struck parton is directly connected to the incoming quark through a quark line. On the other hand, PS contributions arise from cut diagrams in which, on both sides of the cut, the struck parton is separated from the incoming quark by gluon lines.

Chapter 2

Semi-Inclusive Deep Inelastic Scattering

In this chapter, we provide an overview of SIDIS process, which is the core of the studies of this thesis. The main difference compared to the DIS case is that we also measure one or more hadronic final states. Therefore, we are no longer fully inclusive over the final hadronic states, and the factorization formula in Eq. 1.4.34 requires modifications, because we need to introduce the fragmentation functions.

In particular, we briefly introduce the kinematic variables, which closely resemble those used in DIS case. We then present the SIDIS differential cross-section, the fragmentation functions and their evolution and the SIDIS factorization formula. Furthermore, we also introduce the kinematic variables for the Drell-Yan process at fixed rapidity.

Secondly, we study the nature of the enhanced logarithms that emerge in the SIDIS process when the scaling variables become large, namely in the so-called double- and single-soft limits, which are the regimes of interest in this thesis. To this end, we analyze the behavior of the coefficient function in these limits. In particular, we extract its dimensional dependence in terms of powers of a soft scale by studying the phase space and the partonic amplitudes. As a result, we derive the form of the enhanced logarithms, which appear at all orders in the coefficient function in the soft limits and necessitate a resummation procedure, the main topic of the next chapter.

We observe that from the phase space investigation, we obtain one of the main features of this section: the SIDIS phase space correspondence with its crossed version, the Drell-Yan process, as discussed in section 2.2.1. We also note that this study directly provides the resummation formula for the SIDIS process, presented in section 3.

2.1 Kinematics

2.1.1 SIDIS Kinematic variables

We consider the SIDIS $l(k) + H_1(P_1) \to l(k') + H_2(P_2)$ (Fig. 2.1) with momentum transfer $q = k - k'$, hence:

$$H_1(P_1) + \gamma^*(q) \to H_2(P_2) + X \ . \tag{2.1.1}$$

H_1 represents the incoming hadron state, and H_2 is the outgoing measured hadron state. γ^* is the virtual gauge boson which mediates the interaction between the incoming hadron and lepton. The variable X denotes the remaining hadronic radiation. In particular, at the parton level, the variables ξ_1 and ξ_2 in Fig.2.1 and Fig.2.2 represent, respectively, the momentum fraction of the initial hadron carried by the incoming parton and the momentum fraction that the produced hadron takes from the parent parton.

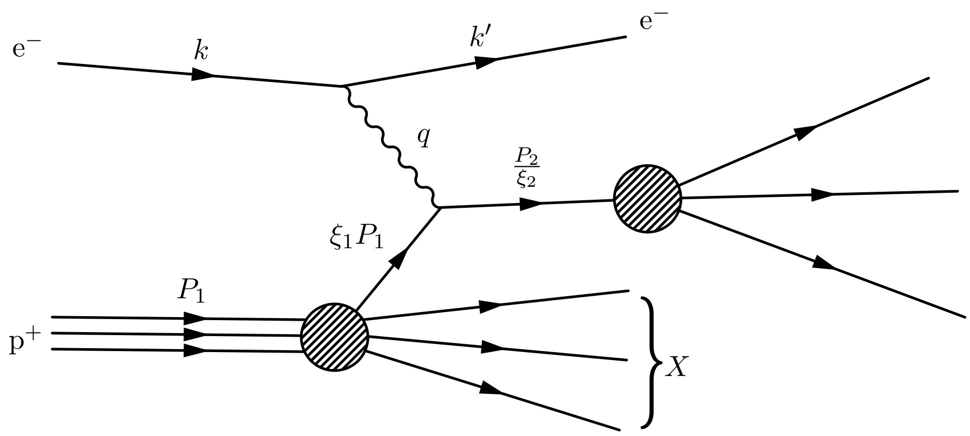


Figure 2.1: SIDIS process

The hadronic process is described by the kinematic variables:

$$Q^2 = -q^2 = -(k - k')^2\,, \tag{2.1.2}$$

$$x = \frac{Q^2}{2P_1 \cdot q} \quad \text{space-like variable,} \tag{2.1.3}$$

$$z = \frac{P_1 \cdot P_2}{P_1 \cdot q} \quad \text{time-like variable,} \tag{2.1.4}$$

$$y = \frac{P_1 \cdot q}{P_1 \cdot k}\,. \tag{2.1.5}$$

Where, $Q^2 = xy\tilde{s}$ and $\sqrt{\tilde{s}}$ is the center-of-mass (c.m.) energy for the incoming lepton and nucleon. If we consider only the electromagnetic interaction, in addition to the structure functions' dependence on the scaling variable z, and the terms that cancel after integrating over the azimuthal angle of the outgoing hadron, one obtains the usual DIS tensor in Eq. 1.3.26 [18]:

$$\begin{aligned} W^{\mu\nu}(P,q) = &\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1^{H_2}(x,z,Q^2) \\ &+ \left(P_1^\mu - \frac{P_1 \cdot q}{q^2} q^\mu\right)\left(P_1^\nu - \frac{P_1 \cdot q}{q^2} q_\nu\right) \frac{F_2^{H_2}(x,z,Q^2)}{P \cdot q}\,. \end{aligned} \tag{2.1.6}$$

Therefore, the triple-differential cross section may be written as [19] [20]

$$\frac{d\sigma^{H_2}}{dxdydz} = \frac{4\pi\alpha^2}{Q^2}\left[\frac{1+(1-y)^2}{2y}\mathcal{F}_T^{H_2}(x,z,Q^2) + \frac{1-y}{y}\mathcal{F}_L^{H_2}(x,z,Q^2)\right]. \tag{2.1.7}$$

Here, α is the fine structure constant and $\mathcal{F}_L^h \equiv \frac{F_L^h}{x}$ and $\mathcal{F}_T^h \equiv 2F_1^h$ are longitudinal and transverse structure functions. We note that a part the structure functions dependence by the scaling variable z this cross section is analogous to the one obtained in the DIS case in Eq. 1.3.28.

At the partonic level, the SIDIS process reads as follows:

$$f_1(p_1) + \gamma^*(q) \to f_2(p_2) + X\;, \tag{2.1.8}$$

f_1 is the incoming parton from the parent hadron H_1. f_2 is the outgoing parton that fragments to the outgoing hadron H_2. Hence, $p_1 = \xi_1 P_1$ and $p_2 = \frac{P_2}{\xi_2}$, where ξ_1, ξ_2 are momentum fractions, then $0 \leq \xi_1, \xi_2 \leq 1$. γ^* is the virtual gauge boson and X is the remaining parton radiation. Therefore, the partonic kinematic variables become:

$$\hat{x} = \frac{Q^2}{p_1 \cdot q} = \frac{Q^2}{\xi_1 P_1 \cdot q} = \frac{x}{\xi_1}, \tag{2.1.9}$$

$$\hat{z} = \frac{p_1 \cdot p_2}{p_1 \cdot q} = \frac{\xi_1 P_1 \cdot \frac{P_2}{\xi_2}}{\xi_1 P_1 \cdot q} = \frac{z}{\xi_2} \ . \tag{2.1.10}$$

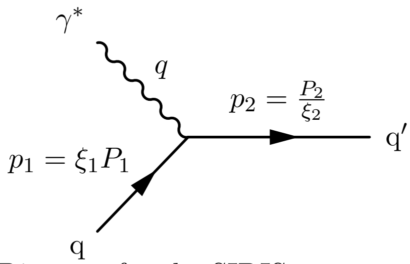


Figure 2.2: Diagram for the SIDIS process at leading order.

At the leading order, the partonic process is represented in Fig. 2.2. Then from momentum conservation is interesting to observe that

$$z = \frac{P_1 \cdot P_2}{P_1 \cdot q} = \frac{p_1 \cdot \xi_2 p_2}{p_1 \cdot q} = \xi_2 \frac{p_1 \cdot (p_1 + q)}{p_1 \cdot q} = \xi_2 \rightarrow z = \xi_2 \,, \tag{2.1.11}$$

$$(p_1 + q)^2 = (\xi_1 P_1 + q)^2 = 2\xi_1 P_1 \cdot q - Q^2 = 0 \rightarrow \xi_1 = x \,, \tag{2.1.12}$$

In both cases, we have used the massless condition for the partons. Thus, at leading order, x represents the momentum fraction carried by the incoming parton relative to the incoming proton, while z corresponds to the momentum fraction carried by the outgoing hadron relative to the outgoing quark.

2.1.2 DY process at fixed rapidity

In this section, we briefly review the kinematics of the DY process at fixed rapidity in order to introduce its correspondence with the SIDIS process in the threshold limit. To this aim we follow, the arguments presented in [3].

The DY process is given by the interaction of two hadrons into a lepton pair, namely

$$H_1(P_1) + H_2(P_2) \rightarrow l(k) + l'(k') + X \tag{2.1.13}$$

where H_1 and H_2 are the incoming hadrons and l, l' are the outgoing leptons. As for the SIDIS case we can consider the interaction mediated by a virtual photon. Therefore, at the parton level, the interaction is given by

$$f_1(p_1) + f_2(p_2) \rightarrow \gamma^*(p) + X(k) \,, \tag{2.1.14}$$

f_1 is the incoming parton from the parent hadron H_1 and f_2 is the incoming parton from the parent hadron H_2. At the leading order the process is represented in Fig. 2.3 what one immediately can see from the diagram in Fig. 2.3 is the fact that DY process is the crossed version of the SIDIS case. After this consideration, we can introduce its kinematic. The DY partonic cross

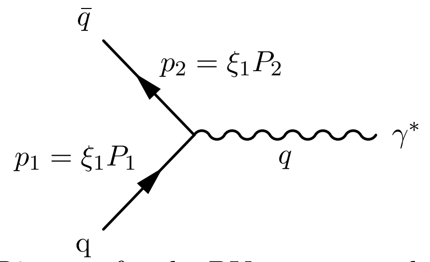


Figure 2.3: Diagram for the DY process at the leading order.

section is expressed through the rapidity of the outgoing gauge boson, namely its longitudinal boost, and the following scaling variable

$$\tau \equiv \frac{M^2}{s} \quad s = (p_1 + p_2)^2 \tag{2.1.15}$$

where M^2 is the invariant mass of the final lepton state. We now consider the system in the partonic center of mass (CM) frame, then we obtain

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1) \tag{2.1.16}$$

$$p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1) \tag{2.1.17}$$

$$p_1 + p_2 = \sqrt{s}(1,0,0,0) \tag{2.1.18}$$

$$p = \left(\sqrt{M^2 + p_t^2 + p_z^2}, \vec{p}_t, p_z\right) = \left(\sqrt{M^2 + p_z^2}\cosh y, \vec{p}_t, \sqrt{M^2 + p_z^2}\sinh y\right) \tag{2.1.19}$$

$$k = \left(\sqrt{M_X^2 + p_t^2 + p_z^2}, -\vec{p}_t, -p_z\right) \tag{2.1.20}$$

then we want re-express the dependence of the process through two new scaling variables

$$x_1 = \sqrt{\tau}e^{y}\,, \tag{2.1.21}$$
$$x_2 = \sqrt{\tau}e^{-y}. \tag{2.1.22}$$

so that

$$x_1 x_2 = \tau \tag{2.1.23}$$

$$y = \frac{1}{2}\ln\frac{x_1}{x_2} \tag{2.1.24}$$

in particular the Jacobian of the transformation (x_1, x_2) to (y, τ) equals one. As it shown in Sec. 2.2.1, the above changing variable makes clear the correspondence between the soft limits in the DY and SIDIS processes. Moreover, we note that, for fixed τ, the rapidity domain is constrained by the fact that $|p_z|$ in Eq. 2.1.19 cannot exceed the value allowed by the maximum available energy. We now note that s is given by

$$s = (p_1+p_2)^2 = (q+k)^2 = M^2 + M_X^2 + 2p_t^2 + 2p_z^2 + 2\sqrt{M^2 + p_t^2 + p_z^2}\sqrt{M_x^2 + p_t^2 + p_z^2}\,, \tag{2.1.25}$$

then at fixed rapidity (at fixed p_z), we can note that the minimum value of s corresponds to the minimum energy configuration, determined by both the photon's energy and the invariant mass

M_x of the recoiling system X. Therefore, the minimum value of the energy s is reached when $p_t = 0$ and $M_X = 0$, thus

$$s \geq s_{\min}(p_z) = \left(\sqrt{M^2 + p_z^2} + \sqrt{p_z^2}\right) . \tag{2.1.26}$$

Therefore, we obtain that

$$p_z \leq p_{z_{\max}} = \frac{s - M^2}{2\sqrt{s}} = \frac{1-\tau}{2\sqrt{\tau}} , \tag{2.1.27}$$

then $p_{z_{\max}}$ is reached when $p_t = 0$ then using Eq. 2.1.19 we have $p_{z_{\max}} = M \sinh y^{\max}$, hence

$$\ln \tau \leq y \leq \ln \frac{1}{\sqrt{\tau}}. \tag{2.1.28}$$

The corresponding bounds on x_1 and x_2 are

$$0 \leq x_1, x_2 \leq 1. \tag{2.1.29}$$

Using the Eq. 2.1.19 the double-soft limit is defined as follows

$$s \to s_{\min}(0) = M^2 \tag{2.1.30}$$

that means

$$p_t \to 0 \quad p_z \to 0 \Rightarrow \tau \to 1 \quad y \to 0 \tag{2.1.31}$$

namely

$$x_1 \to 1 \quad x_2 \to 1 \, . \tag{2.1.32}$$

On the other hand single-soft limit is defined as

$$s \to s_{\min}(p_z) \tag{2.1.33}$$

at fixed p_z namely at fixed rapidity. Therefore, in terms of variables x_i Eq. 2.1.21 the condition of fixed y implies fixed ratio $\frac{x_1}{x_2} = e^{2y}$ and the condition of minimum s implies that τ must be at its maximum, i.e. maximum product $x_1 x_2$. If we assume without loss of generality $x_1 \geq x_2$, for fixed e^{2y} we have maximum $x_1 e^{-2y}$, namely $x_1 = 1$. So the single-soft is the limit where either

$$x_1 \to 1 \quad x_2 \text{ fixed} \tag{2.1.34}$$

or

$$x_2 \to 1 \quad x_1 \text{ fixed} \tag{2.1.35}$$

2.1.3 Fragmentation functions

In the SIDIS process, compared to the DIS case, we need to account for a new non-perturbative effect: the probability that the outgoing parton fragments into the measured final hadron. This probability is defined through the fragmentation function (FF), referred to as $D_j^H(\xi_2, \mu_{\mathrm{F}})$, which represents the probability that the parton j, with a longitudinal momentum fraction P/ξ, fragments into the observed hadron H. In analogy to the PDF case the evolution of the FFs is predicted by the DGLAP equation [21]. Hence, in analogy to Eq. 1.5.5, we obtain

$$\mu_{\mathrm{F}}^2 \frac{\partial D_j^H(z, \mu_{\mathrm{F}})}{\partial \mu_{\mathrm{F}}^2} = \frac{\alpha_s(\mu_{\mathrm{R}})}{2\pi} \sum_j P_{ij}^T \otimes D_j^H(z, \mu_{\mathrm{F}}) \, , \tag{2.1.36}$$

where $\mu_{\rm F}$ is the factorization scale, the sum runs over all possible partons j, while i represents a specific parton, and P^T_{ij} denotes the time-like splitting functions. These, compared to the space-like splitting functions, are obtained from gluon emissions off the outgoing parton line, namely in the case where the emission occurs after the hard interaction with the photon. At leading order, there is no difference between time-like and space-like contributions. However, beyond leading order, virtual contributions appear, and the direction of the Wick rotation differs, resulting in sign differences between the space-like and time-like cases. In particular, we note the difference compared to the PDF case. In that case, the kinematics of the incoming parton is governed by the Bjorken variable, making the kinematics space-like, and we thus use space-like splitting functions, P_{ij}, in order to evolve the PDFs. On the other hand, the kinematics of the outgoing parton is governed by the z variable, which is time-like, so we need to use time-like splitting functions, P^T_{ij}, in order to evolve the FFs.

As we shown in sec. 1.5.1, in order to decouple the $2N_f + 1$ DGLAP equations. We can split the FFs into flavour singlet and NS contributions. Hence, we define the singlet term as

$$D^H_S \equiv \frac{1}{N_f}\sum_i^{N_f}(D^H_{q_i} + D^H_{\bar{q}_i})\,, \tag{2.1.37}$$

which evolves together with D^H_g according to

$$\frac{\partial}{\partial Q^2}\begin{pmatrix} D^H_S \\ D^H_g \end{pmatrix} = \begin{pmatrix} P^T_{qq} & P^T_{qg} \\ P^T_{gq} & P^T_{gg} \end{pmatrix} \otimes \begin{pmatrix} D^H_S \\ D^H_G \end{pmatrix}\,, \tag{2.1.38}$$

while the NS sector is given by three non-singlet combinations of PDFs, namely

$$D^{H,\pm}_{{\rm NS},ik} \equiv D^H_{q_i} \pm D^H_{\bar{q}_i} - (D^H_{q_k} \pm D^H_{\bar{q}_k}) \tag{2.1.39}$$

$$D^{V,H}_{\rm NS} \equiv D^H_{q_i} - D^H_{\bar{q}_i} \tag{2.1.40}$$

which evolve independently with $P^{T,+}_{\rm NS}$, $P^{T,-}_{\rm NS}$, $P^{T,V}_{\rm NS}$ and decouple the remaining $2N_f - 1$ equations. Specifically, we have

$$P^{\pm}_{T,{\rm NS}} = P^{T,V}_{qq} \pm P^{T,V}_{q\bar{q}} \tag{2.1.41}$$

$$P^{T,V}_{\rm NS} = P^{T,V}_{qq} - P^{V,T}_{q\bar{q}} + N_f(P^{S,T}_{qq} - P^{T,V}_{q\bar{q}}) \equiv P^{T,-}_{\rm NS} + P^{T,S}_{\rm NS}\,, \tag{2.1.42}$$

where we have rewritten the time-like splitting functions in terms of flavour singlet (S) and non-singlet (V) as

$$P^T_{q_iq_j} = \delta_{ij}P^{T,V}_{qq} + P^{T,S}_{qq}\,, \tag{2.1.43}$$

$$P^T_{q_i\bar{q}_j} = \delta_{ij}P^{T,V}_{q\bar{q}} + P^{T,S}_{q\bar{q}}\,. \tag{2.1.44}$$

All the NS and PS time-like splitting functions are known up to NNLO [22–24].

2.1.4 z-distribution factorization

In section 1.4, we apply the factorization theorem to express the hadronic cross section as a convolution of a long-scale perturbative term and a short-scale non-perturbative term, specifically we obtained the Eq. 1.4.34. Although a rigorous proof of the factorization theorem via the Operator Product Expansion (OPE) cannot be provided for the SIDIS case, alternative methods have been developed to establish it (see, for instance, [6]). Therefore, we can state that the factorization theorem holds for SIDIS as well. Due to the different nature of the two non-perturbative phenomena present in the SIDIS process, described by the PDF and the FF, the SIDIS $z-$differential hadronic cross-section can be factorized as the convolution of a $\hat{z}-$differential partonic cross-section with a PDF and a FF:

$$\frac{d\sigma}{dz}(x,z,Q^2) = \sum_{i,j} \int_x^1 \frac{\mathrm{d}\xi_1}{\xi_1} \int_z^1 \frac{\mathrm{d}\xi_2}{\xi_2} f_i(\xi_1) D_j^{H_2}(\xi_2) \frac{d\hat{\sigma}_{ij}}{d\hat{z}} \left(\frac{x}{\xi_1}, \frac{z}{\xi_2}\right) \ . \tag{2.1.45}$$

Here, $f_i(\xi_1)$ is the PDF of parton $i = q, \bar{q}, g$ in the nucleon at momentum fraction ξ_1, while $D_j^{H_2}(\xi_2)$ is the corresponding FF for the parton j going to the observed hadron H_2. The functions $\frac{d\hat{\sigma}_{ij}}{d\hat{z}}(\hat{x}, \hat{z})$ are the partonic differential cross sections for incoming parton i and outgoing parton j and represent the perturbative terms of the convolution. Finally, note that, to simplify the notation, we omit the dependence on the renormalization and factorization scales, μ_R and μ_F, respectively.

After the change of variables, we can express the z distribution as follows:

$$\frac{d\sigma}{dz}(x,z,Q^2) = \sum_{i,j} \int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}} \int_z^1 \frac{\mathrm{d}\hat{z}}{\hat{z}} f_i\left(\frac{x}{\hat{x}}\right) D_j^{H_2}\left(\frac{z}{\hat{z}}\right) \frac{d\hat{\sigma}_{ij}}{d\hat{z}}(\hat{x}, \hat{z}) \ . \tag{2.1.46}$$

Secondly, we express the z-distribution in Mellin space, as the resummation procedure showed in the next chapters cannot be performed in $\hat{x}, \hat{z}$ space (see, for instance, [25]); while for the definition and properties of the Mellin transformation the interest reader can find a summary about it in Appendix A.2. For simplicity, with a little abuse of notation we denote $z-$distributions only by their dependences, namely:

$$\frac{d\sigma}{dz}(x,z,Q^2) := \sigma(x,z,Q^2) \ , \tag{2.1.47}$$

$$\frac{d\hat{\sigma}_{ij}}{dz}(x,z,Q^2) := \hat{\sigma}_{ij}(x,z,Q^2) \ . \tag{2.1.48}$$

Applying a double-Mellin transformation to Eq.2.1.46, we obtain:

$$\tilde{\sigma}(N,M,Q^2) = \int_0^1 dx x^{N-1} \int_0^1 dz z^{M-1} \sigma(x,z,Q^2) \ , \tag{2.1.49}$$

$$\tilde{\sigma}(N,M,Q^2) = \sum_{i,j} \tilde{f}_i(N) \tilde{D}_j^{H_2}(M) \tilde{\hat{\sigma}}_{ij}(N,M,Q^2) \ , \tag{2.1.50}$$

where

$$\tilde{f}_i(N) = \int_0^1 dx x^{N-1} f_i(x) \ , \tag{2.1.51}$$

$$\tilde{D}_j^{H_2}(M) = \int_0^1 dz z^{M-1} D_j^{H_2}(z) \ , \tag{2.1.52}$$

$$\tilde{\hat{\sigma}}_{ij}(N,M,Q^2) = \int_0^1 d\hat{x} \hat{x}^{N-1} \int_0^1 d\hat{z} \hat{z}^{M-1} \ \hat{\sigma}_{ij}(x,z,Q^2) \ . \tag{2.1.53}$$

To obtain the above relations we used the factorisation of convolution product under a Mellin transformation A.2. Furthermore, we stress that the above argument can also be applied to the individual terms of the SIDIS cross section in Eq. (2.1.7), namely the longitudinal and transverse structure functions. Owing to the universality of the parton density and the fragmentation function, it suffices to split the perturbative partonic cross section into its transverse and longitudinal components. Therefore, using $H_2 = h$ for notation simplicity, the structure functions are expressed as

$$\mathcal{F}_k^h(x,z,Q^2) = \sum_{ij} \int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}} \int_z^1 \frac{\mathrm{d}\hat{z}}{\hat{z}} f_i\left(\frac{x}{\hat{x}}, \mu_{\mathrm{F}}\right) C_k^{ij}\left(\hat{x}, \hat{z}, \alpha_s(\mu_{\mathrm{R}}), \frac{\mu_{\mathrm{R}}}{Q}, \frac{\mu_{\mathrm{F}}}{Q}\right) D_j^h\left(\frac{z}{\hat{z}}, \mu_{\mathrm{F}}\right) \quad (2.1.54)$$

where $k = T, L$ and C^k are the coefficient functions. So in Mellin space the structure functions become

$$\tilde{\mathcal{F}}_k^h(N,M,Q^2) = \sum_{ij} \tilde{f}_i(N,\mu_{\mathrm{F}}) \tilde{C}_k^{ij}\left(N, M, \alpha_s(\mu_{\mathrm{R}}), \frac{\mu_{\mathrm{R}}}{Q}, \frac{\mu_{\mathrm{F}}}{Q}\right) \tilde{D}_j^{H_2}(M,\mu_{\mathrm{F}}) \,, \quad (2.1.55)$$

with $\tilde{f}_i$ and $\tilde{D}_j^h$ defined as above and

$$\tilde{C}_k^{ij}\left(N, M, \alpha_s(\mu_{\mathrm{R}}), \frac{\mu_{\mathrm{R}}}{Q}, \frac{\mu_{\mathrm{F}}}{Q}\right) = \int_0^1 d\hat{x}\hat{x}^{N-1} \int_0^1 d\hat{z}\hat{z}^{M-1}\, C_k^{ij}\left(\hat{x}, \hat{z}, \alpha_s(\mu_{\mathrm{R}}), \frac{\mu_{\mathrm{R}}}{Q}, \frac{\mu_{\mathrm{F}}}{Q}\right) . \quad (2.1.56)$$

As it shown for the DIS case similarly the coefficient function of the SIDIS process can be rewritten in NS and PS terms, for instance at NNLO as it done in [26] and [21] we have

$$C_{p'p}^{i} = C_{p'p}^{i,(0)} + \frac{\alpha_s(\mu_R^2)}{\pi} C_{p'p}^{i,(1)} + \left(\frac{\alpha_s(\mu_R^2)}{\pi}\right)^2 C_{p'p}^{i,(2)} + \mathcal{O}(\alpha_s^3) \,. \quad (2.1.57)$$

where $i = T, L$. Therefore, using the notation NS and PS we have

$$\begin{aligned}
C_{qq}^{i,(2)} &= C_{\bar{q}\bar{q}}^{i,(2)} = e_q^2 C_{qq}^{i,\mathrm{NS}} + \left(\sum_j e_{q_j}^2\right) C_{qq}^{i,\mathrm{PS}} \,, \\
C_{\bar{q}q}^{i,(2)} &= C_{q\bar{q}}^{i,(2)} = e_q^2 C_{\bar{q}q}^{i} \,, \\
C_{q'q}^{i,(2)} &= C_{\bar{q}'\bar{q}}^{i,(2)} = e_q^2 C_{q'q}^{i,1} + e_{q'}^2 C_{q'q}^{i,2} + e_q e_{q'} C_{q'q}^{i,3} \,, \\
C_{\bar{q}'q}^{i,(2)} &= C_{q'\bar{q}}^{i,(2)} = e_q^2 C_{q'q}^{i,1} + e_{q'}^2 C_{q'q}^{i,2} - e_q e_{q'} C_{q'q}^{i,3} \,, \\
C_{gq}^{i,(2)} &= C_{g\bar{q}}^{i,(2)} = e_q^2 C_{gq}^{i} \,, \\
C_{qg}^{i,(2)} &= C_{\bar{q}g}^{i,(2)} = e_q^2 C_{qg}^{i} \,, \\
C_{gg}^{i,(2)} &= \left(\sum_j e_{q_j}^2\right) C_{gg}^{i} \,,
\end{aligned} \quad (2.1.58)$$

again for $i = T, L$. With q' ($\bar{q}'$) we indicate a quark (antiquark) of flavour different from q, whereas the NS and PS superscripts in the quark-to-quark channel denote the non-singlet and the pure-singlet components respectively.

2.2 Soft limits

As shown in the previous chapter for the DIS case, starting from NLO we have real gluon emissions either from the incoming quark line, the outgoing quark line, or both. The aim of this thesis is to to study what happen when the radiation reaches the boundaries of the phase space. In particular this is achieved in the following cases:

- The **double-soft limit**, where $\hat{x}$, $\hat{z} \to 1$.
- The **single-soft limit**, also referred to as the asymmetric case, where either $\hat{x} \to 1$ or $\hat{z} \to 1$ while the other scaling variable remains fixed.

Therefore, we have to solve three problems. Firstly, we need to show to which system configurations these limits correspond. Secondly, we need to understand which kinematical configurations of the outgoing radiation correspond to the three soft limits. Finally, we must analyze how the differential partonic cross section behaves in these limits, implying the need for a resummation procedure.

2.2.1 Phase space in soft limits

We now analyze the phase space of the partonic process in Eq. (2.1.8) in soft limits in order to extract the dependence of the partonic cross-section, and consequently the coefficient functions, in the double- and single-soft limits through the scaling variables $\hat{x}$ and $\hat{z}$.

We focus on the $\hat{z}-$distribution when the system X includes n final-state massless partons with momenta $k_1, \ldots, k_n$, which are emitted from both the incoming (p_1) and the outgoing (p) partons. The momenta of the incoming and outgoing partons must satisfy the momentum conservation equation:

$$p_1 + q = p + k_1 + ... + k_n \ . \tag{2.2.1}$$

We are interested in the double- and single-soft limits and in how the SIDIS phase space behaves in this limit similar to the double- and single soft limits of the Drell-Yan process in rapidity, [3] and Sec. 2.1.2. This correspondence allows us to apply the same resummation formula to both processes.

To this end, we use the partonic Breit frame. The frame is defined by the gauge boson and incoming parton being back-to-back, with q being purely spatial, this last choice is possible since q is a space-like vector. Defining the four-momentum of the virtual gauge boson as

$$q = (0, 0, 0, -Q) \ , \tag{2.2.2}$$

and imposing the conditions $s = (p_1 + q)^2$, that p_1 and q are back-to-back, and that $p_1^2 = 0$, we obtain:

$$p_1 = \frac{s + Q^2}{2Q}(1, 0, 0, 1) \ . \tag{2.2.3}$$

In order to obtain the $\hat{z}$ dependence for p, we use the fact that $p_1 \cdot p = \hat{z} p_1 \cdot q$ and $p^2 = 0$, leading to:

$$p = \left(\frac{1}{2}\left(\frac{p_t^2}{\hat{z}Q} + \hat{z}Q \right), \vec{p}_t, \frac{1}{2}\left(\frac{p_t^2}{\hat{z}Q} - \hat{z}Q \right) \right) \ . \tag{2.2.4}$$

We now can use the crossing symmetry between SIDIS and DY process. In the DY case (Sec. 2.1.2) the single-soft limits are determined through the Mandelstam variable $s = (p_1 + p_2)^2$. Then, using the notation adopted in the two cases and exploiting the crossing symmetry, we have the following correspondences

$$\text{DY} \quad p_1 \leftrightarrow p_1 \quad \text{SIDIS} \tag{2.2.5}$$
$$\text{DY} \quad p_2 \leftrightarrow -p \quad \text{SIDIS} \tag{2.2.6}$$
$$\text{DY} \quad k \leftrightarrow k \quad \text{SIDIS} \tag{2.2.7}$$
$$\text{DY} \quad p \leftrightarrow -q \quad \text{SIDIS} \tag{2.2.8}$$

therefore

$$\text{DY} \quad s = (p_1 + p_2)^2 \leftrightarrow t = (p_1 - p)^2 = -\frac{\hat{z}}{\hat{x}} Q^2 \quad \text{SIDIS} \, . \tag{2.2.9}$$

In the DY case in the soft limits s is at its minimum value, therefore also t in the SIDIS case in the soft-limits must be at its minimum value.

The double-soft limit in the DY case is provided by the condition $s = M^2$, hence in its crossed version (SIDIS case) is obtained by $t = -Q^2$ with $\hat{x}, \hat{z} \to 1$, which is the so-called elastic limit. Whereas, in the DY process single-soft limits correspond to the case where the photon has a fixed longitudinal momentum with s at its minimum (then $k^2 = p_t = 0$). Thus, the photon must have fixed longitudinal momentum even in the single-soft limits of the crossed process. We now observe that in the SIDIS process we have

$$k^2 = (p_1 + q - p)^2 \tag{2.2.10}$$

and using the Breit frame we obtain:

$$k^2 = s(1-\hat{z}) - \frac{p_t^2}{\hat{z}} \ , \tag{2.2.11}$$

thus

$$k^2 \leq s(1-\hat{z}) = Q^2 \frac{(1-\hat{x})(1-\hat{z})}{\hat{x}} = k^2_{max} \ . \tag{2.2.12}$$

Here, we use the fact that $s = Q^2 \frac{1-\hat{x}}{\hat{x}}$. Therefore, in the single soft limits $k^2 \to 0$ that implies $p_t \to 0$. Then, using Eqs. 2.2.4 and 2.2.3, we have

$$p_1 = \frac{1}{2}(Q, 0, 0, Q) \quad \hat{x} \to 1 \, , \tag{2.2.13}$$

$$p = \frac{1}{2}(Q, 0, 0, -Q) \quad \hat{z} \to 1 \tag{2.2.14}$$

then in the single-soft limits one of the two above conditions is satisfied. That implies that either the incoming parton p_1 or the outgoing parton p has fixed longitudinal momentum, because Q is fixed. Furthermore, in this cases the exchange of momentum t approaches its minimum value in order to allow this configuration.

As a final step, we determine the relation which determines the kinematical configurations of the extra radiation in the soft-limits. Using the fact that all partons are massless, squaring Eq.2.2.1 we obtain:

$$s = \sum_{i,j=1}^{n} k_i \cdot k_j + 2 \sum_{i=1}^{n} p \cdot k_i \ , \tag{2.2.15}$$

where both terms in the LHS are positive semi-definite. Moreover, in Breit frame it becomes

$$Q^2 \frac{1-\hat{x}}{\hat{x}} = \sum_{i,j=1}^{n} k_i^0 k_j^0 (1 - \cos\theta_{ij}) + 2 \sum_{i=1}^{n} \frac{1}{2} \left(\frac{p_t^2}{\hat{z}Q} + \hat{z}Q \right) k_i^0 (1 - \cos\theta_{i2}). \tag{2.2.16}$$

One of the aim of the next section is to exploit the above relations to obtain the kinematical configuration of the outgoing partons in the threshold limits.

2.2.2 Double-soft limit

As we can see in the appendix of [25] we can write the phase space as:

$$d\phi_{n+1}(p_1 + q;\, p,\, k_1, \ldots, k_2) = \int \frac{dk^2}{2\pi}\, d\phi_2(p_1 + q;\, p,\, k)\, d\phi_n(k;\, k_1, \ldots, k_n) \tag{2.2.17}$$

with $n \geq 0$. Here, $d\phi_2$ is the phase space for production from the incoming total momentum $p_1 + q$ of a massless final state with momentum p differential in $\hat{z}$, and system with momentum k recoiling against it. Meanwhile, $d\phi_n$ represents the phase space for the production, from the incoming momentum k, of a n-parton final state with momenta k_i.

We now compute the two-body phase space in terms of the $\hat{z}$ variable in $d = 4-2\epsilon$ dimensions. In the Breit frame we have:

$$\begin{aligned} d\phi_2(p_1+q;p,k) &= \frac{d^{d-1}k}{(2\pi)^{d-1}2k^0}\frac{d^{d-1}p}{(2\pi)^{d-1}2p^0}(2\pi)^d\ \delta^{(d)}\left(p_1+q-p-k\right) \\ &= \frac{(2\pi)^{d-2}}{4}\frac{d^{d-1}p}{p^0k^0}\ \delta\left(p_1^0+q^0-p^0-k^0\right) \\ &= \frac{(4\pi)^{\epsilon}}{16\pi\Gamma(1-\epsilon)}\frac{(p_t^2)^{-\epsilon}}{p^0k^0}dp_t^2dp_{2_z}\ \delta\left(\frac{s+Q^2}{2Q}-p^0-k^0\right)\ . \end{aligned} \tag{2.2.18}$$

In the second equality, we perform the integration using the three-dimensional delta function. In the third equality, we express the integration over $d^{d-2}p_t$ in spherical coordinates and exploit the identity for the solid angle $\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$, where $\Gamma(x)$ denotes the Gamma function. For notation simplicity, from this point onward, we will identify p as p.

Next, we aim to express the phase space as differential in $\hat{z}$. Therefore, using 2.2.4, we perform the following change of variables:

$$p_z = \frac{1}{2}\left(\frac{p_t^2}{\hat{z}Q}-\hat{z}Q\right) \tag{2.2.19}$$

therefore

$$dp_z = \frac{dp_z}{d\hat{z}}d\hat{z} = -\frac{1}{\hat{z}}\frac{1}{2}\left(\frac{p_t^2}{\hat{z}Q}+\hat{z}Q\right)d\hat{z} = -\frac{1}{\hat{z}}p^0d\hat{z}. \tag{2.2.20}$$

Thus, Eq.2.2.18 is equal to:

$$-\frac{(4\pi)^{\epsilon}}{16\pi\Gamma(1-\epsilon)}\frac{(p_t^2)^{-\epsilon}}{\hat{z}k^0}dp_t^2d\hat{z}\ \delta\left(\frac{s+Q^2}{2Q}-p^0-k^0\right). \tag{2.2.21}$$

We now perform the integration over p_t^2. To this end, we need to express the argument of the δ-function in terms of a function of p_t^2. We note that, by momentum conservation, $\vec{k_t} = -\vec{p_t}$, so that $|\vec{k_t}| = |\vec{p_t}| = p_t$. Hence, we have:

$$k^0 = \sqrt{k^2+p_t^2+(k^3)^2}\ .$$

Furthermore, by expressing k^3 as $p_1^3+q^3-p^3$, and using the fact from Eq.2.2.4 that p^0 and p^3 can be expressed in terms of p_t^2, we find that the argument of the δ-function becomes:

$$\begin{aligned} g(p_t^2) &= \frac{s+Q^2}{2Q}-p^0-\sqrt{k^2+p_t^2+\left(\frac{s-Q^2}{2Q}\right)^2+(p^3)^2-2\left(\frac{s-Q^2}{2Q}\right)p^3} \\ &= \frac{s+Q^2}{2Q}-p^0-\sqrt{k^2+(p^0)^2+\left(\frac{s-Q^2}{2Q}\right)^2-2\left(\frac{s-Q^2}{2Q}\right)p^3}. \end{aligned} \tag{2.2.22}$$

In the second equality we use the fact that $p^2 = 0$. Hence, the solution to the equation $g(p_t^2) = 0$ is $\tilde{p_t} = \hat{z}[s(1-\hat{z}) - k^2]$. Furthermore, using from Eq.2.2.4 the fact that $\frac{dp^3}{dp_t^2} = \frac{dp^0}{dp_t^2}$, we obtain:

$$\begin{aligned}\left|\frac{dg(p_t^2)}{dp_t^2}\right| &= \left[1 + \frac{1}{k^0}\left(p^0 - \frac{s-Q^2}{2Q}\right)\right]\frac{dp^0}{dp_t^2} \\ &= \left[1 + \frac{1}{k^0}\left(\frac{s+Q^2}{2Q} - k^0 - \frac{s-Q^2}{2Q}\right)\right]\frac{dp^0}{dp_t^2} \\ &= \frac{Q}{k^0}\frac{dp^0}{dp_t^2} = \frac{1}{2\hat{z}k^0}.\end{aligned} \tag{2.2.23}$$

Thus, using the delta identity $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{\left|\frac{df(x)}{dx}|_{x=x_i}\right|}$, we can rewrite the 2-phase space as:

$$d\phi_2(p_1+q;p,k) = -\frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)}(\tilde{p_t}^2)^{-\epsilon}d\hat{z}\ . \tag{2.2.24}$$

Hence, the phase space in Eq.2.2.17 becomes:

$$d\phi_{n+1}(p_1+q;p,k_1,\ldots,k_n) = -d\hat{z}\frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)}\int\frac{dk^2}{2\pi}(\tilde{p_t}^2)^{-\epsilon}d\phi_n(k;k_1,\ldots,k_n). \tag{2.2.25}$$

We can now compute the kinematics limits for k^2 integration. Of course, $k^2_{min} = 0$, e.g. when we have only a single parton in the final state. The upper bound is obtained through the Eq. 2.2.12

Introducing a dimensionless variable v in order to interpolate between 0 and k^2_{max} we can set

$$k^2 = vk^2_{max}; \quad 0 \le v \le 1\ , \tag{2.2.26}$$

and rewrite the measure over k^2 in the phase space Eq.2.2.17 as:

$$dk^2 = dv\ Q^2\frac{(1-\hat{x})(1-\hat{z})}{\hat{x}} \tag{2.2.27}$$

with v ranging from 0 to 1. Furthermore, the phase space $d\phi_n(k;k_1,\ldots,k_n)$ can be viewed as a phase space with the same structure as in deep-inelastic scattering, where the incoming momentum is k^2. Here, the variable k is now integrated over and vanishes in the soft limit. As shown in Ref.[25], it can be written in terms of a dimensionless integration measure, with the dimensional dependence contained in a power of k^2. Thus, using Eq. (4.17) of Ref.[25] we obtain:

$$d\phi_n(k;k_1,\ldots,k_n) = 2\pi\left[\frac{N(\epsilon}{2\pi}\right]^{n-1}(k^2)^{n-2-(n-1)\epsilon}d\Omega^{n-1}(\epsilon), \tag{2.2.28}$$

where $N(\epsilon) = \frac{1}{2(4\pi)^{2-2\epsilon}}$ and

$$d\Omega^{n-1}(\epsilon) = d\Omega_1\ldots d\Omega_{n-1}\int_0^1 dz_{n-1}z_{n-1}^{(n-3)-(n-2)\epsilon}(1-z_{n-1})^{1-2\epsilon}\cdots\int_0^1 dz_2 z_2^{-\epsilon}(1-z_2)^{1-2\epsilon}\ . \tag{2.2.29}$$

The definition of the dimensionless z_i variables is irrelevant here.

We can finally consider the double-soft limit of the phase space. Eq.2.2.12 implies that $k^2 \to 0$ as both $\hat{x}$ and $\hat{z}$ approach 1. Furthermore, as shown in Eq.2.2.28 the dimensional dependence of the phase space $d\phi_n(k;k_1,\ldots,k_n)$ is entirely contained in powers of a soft scale:

$$\Lambda_{ds} = k^2_{max} = Q^2(1-\hat{x})(1-\hat{z})(1+\mathcal{O}(1-\hat{x})+\mathcal{O}(1-\hat{z})). \tag{2.2.30}$$

Hence, from Eq.2.2.25 the double-soft limit dependence on $\hat{x}, \hat{z} \to 1$ for the SIDIS phase space is given by the soft scale in Eq.2.2.30. In other words, since the remain integral of the phase space ranging from 0 to 1, and the SIDIS differential partonic cross-section is a function of the scaling $\hat{x}, \hat{z}$ and the hard scale Q^2 we obtain that

$$\frac{d\hat{\sigma}}{d\hat{z}}(Q^2, \hat{x}, \hat{z}) \underset{\substack{\text{double-soft}\\ \text{limit}}}{\Longrightarrow} \frac{d\hat{\sigma}}{d\hat{z}}(\Lambda_{\rm ds}(Q^2, \hat{x}, \hat{z}))\,, \tag{2.2.31}$$

hence we have just showed what has been anticipated in the sec. 1.4.3.

As it shown in Ref. [3], by replacing the kinematic variables Q^2 with M^2, the invariant mass of the lepton pair, and the pair $\hat{x}$, $\hat{z}$ with x_1, x_2 defined in Sec. 2.1.2, the SIDIS double-soft limit is equivalent to the double-soft limit of its crossed version: the Drell-Yan process at fixed rapidity (DY).

2.2.3 Single-soft limit

We now focus on the single-soft limit, which is very similar to the double-soft limit discussed above. Indeed, the double-soft limit is nothing but a particular case of the single-soft limit. In particular, through the study of this limit we are able to understand the kinematical configuration of the outgoing partons in the soft limits. As we note from Eq. 2.2.17 we have

$$\sum_{i=1}^{n} k_i = k\,. \tag{2.2.32}$$

Instead, from Eqs. 2.2.11 and 2.2.12, in the single-soft limits, namely $\hat{x} \to 1$ and $\hat{z}$ fixed or $\hat{z} \to 1$ and $\hat{x}$ fixed, we obtain the conditions

$$p_t = 0 \to k_t = 0\,, \quad k^2 = 0\,. \tag{2.2.33}$$

Therefore, in the Breit frame as defined above we obtain

$$p = (-p_z, 0, 0, p_z)\,, \tag{2.2.34}$$

$$k = \left(p_z + \frac{s+Q^2}{2Q}, 0, 0, -p_z + \frac{s-Q^2}{2Q}\right)\,. \tag{2.2.35}$$

Hence,

$$2\sum_{i=1}^{n} p\cdot k_i = 2p\cdot k = s\hat{z} = \begin{cases} 0 & \text{if } \hat{x} \to 1\,, \hat{z} \text{ fixed} \\ s & \text{if } \hat{z} \to 1\,, \hat{x} \text{ fixed} \end{cases} \tag{2.2.36}$$

therefore using the momentum conservation relation given by the Eq. 2.2.15 we obtain that in both single-soft limits, so also in the double-soft case, $k_i \cdot k_j = 0$ for all i, j, so all the radiated partons must be collinear. They are not necessarily soft, although some of them may be; the only requirement is that their longitudinal momentum fractions combine in a way that satisfies Eq. 2.2.35.

The phase space can again be written in the form of Eq. 2.2.17. In particular, Eq. 2.2.25 still holds and upper bound of k^2 is still given by k^2_{max} Eq.2.2.12. Therefore, in the single-soft cases by Eq. 2.2.28 we obtain that the dimensional dependence of the phase space is contained in power of a soft scale

$$\Lambda_{ss} = \begin{cases} Q^2(1-\hat{x})\,[1+\mathcal{O}(1-\hat{x})] & \text{if } \hat{x} \to 1\,, \hat{z} \text{ fixed} \\ Q^2(1-\hat{z})\,[1+\mathcal{O}(1-\hat{z})] & \text{if } \hat{z} \to 1\,, \hat{x} \text{ fixed}\,, \end{cases} \tag{2.2.37}$$

therefore

$$\frac{d\hat{\sigma}}{d\hat{z}}(Q^2,\hat{x},\hat{z}) \underset{\substack{\hat{x}-\text{single-soft}\\ \text{limit}}}{\Longrightarrow} \frac{d\hat{\sigma}}{d\hat{z}}(\Lambda_{\rm ss}(Q^2,\hat{x}),\hat{z})\,, \tag{2.2.38}$$

$$\frac{d\hat{\sigma}}{d\hat{z}}(Q^2,\hat{x},\hat{z}) \underset{\substack{\hat{z}-\text{single-soft}\\ \text{limit}}}{\Longrightarrow} \frac{d\hat{\sigma}}{d\hat{z}}(\Lambda_{\rm ss}(Q^2,\hat{z}),\hat{x})\,. \tag{2.2.39}$$

Again, as for the double-soft case, these results are completely equivalent to the ones obtained in the DY at fixed rapidity case [3]. Consequently, it is clear that the kinematic behaviour in the soft limit is the same for the SIDIS and DY processes at fixed rapidity. The only difference lies in the fact that the DY kinematic variables x_1, x_2 are both time-like, whereas in the SIDIS case, $\hat{x}$ is space-like and $\hat{z}$ is time-like. This reflects the fact that one process is the crossed version of the other: indeed, the outgoing quark and the incoming gauge boson in the SIDIS case are exchanged in the DY case.

Chapter 3

Resummation

In this chapter, we provide the resummation formula for the SIDIS case in threshold limit. In particular, we demonstrate how the enhanced logarithms introduced in Sec. 1.4.3 are mapped into Mellin space, then we identify the enhanced logarithms in $\hat{x}$ and $\hat{z}$ space in the threshold limit in the case of the SIDIS process. We then analyze how these terms are organized by the resummation formula in terms of logarithmic towers in the double-soft limit. Consequently, we provide an interpretation of the resummed terms in Mellin space in the single-soft limit. Finally, we derive the resummation formula for a process with single scale dependence, explaining the matching procedure and providing the resummation formula for both soft limits in the SIDIS case.

3.1 Logarithm towers and Mellin space

As shown in Sec. 1.4.3, in the threshold limit, large logarithms in the form of plus distributions appear in the partonic cross section. Specifically, these logarithms dominate the behaviour of the partonic cross section in the soft limit. In the case of a single scale dependence z, which in the threshold (or soft) limit approaches 1, these terms are given by Eq. 1.4.39. Then, they take the form $\alpha_s(\mu)^n \left[\frac{\ln^k(1-\hat{x})}{1-\hat{x}}\right]_+$, where typically $0 \leq k \leq 2n-1$, these logarithms terms are enhanced in the limit $z \to 1$. Threshold resummation accounts for these large logarithmic terms to all orders in the strong coupling. However, as shown in [25], the resummation procedure cannot be performed in z-space. This problem can, however, be resolved by transforming to Mellin space and performing the resummation in that space. Therefore, we briefly review how the plus distributions showed above appear in Mellin space.

As shown in appendix A.2, the Mellin transform exchanges the threshold variable z with a complex with a complex variable N, indeed given a function $f(z)$ its Mellin transformation is

$$\mathcal{M}[f(z)](N) = \int_0^1 \mathrm{d}z\, z^{N-1} f(z)\,. \tag{3.1.1}$$

We note that, in the limit $|N| \to \infty$, the only finite contribution to the integral comes from $f(1)$. Hence, the two limits $z \to 1$ and $|N| \to \infty$ lead to the same result.

So we define,

$$I_p := \int_0^1 \mathrm{d}z\, z^{N-1} \left[\frac{\ln^p(1-z)}{(1-z)}\right]_+ = \int_0^1 \mathrm{d}z \frac{z^{N-1}-1}{1-z} \ln^p(1-z)\,. \tag{3.1.2}$$

Thus, we define the generating functional

$$G(N,\eta) := \int_0^1 \mathrm{d}(z^{N-1}-1)(1-z)^{\eta-1} \tag{3.1.3}$$

we can obtain the plus-distribution term in Mellin space through

$$I_p = \frac{d^p}{d\eta^p} G(N,\eta)\Big|_{\eta=0} \,. \tag{3.1.4}$$

We can recognize the definition of the Euler Beta function in the definition of the generating functional, namely

$$G(N,\eta) = B(N,\eta) - \frac{1}{\eta} = \frac{\Gamma(N)\Gamma(\eta)}{\Gamma(N+\eta)} - \frac{1}{\eta}\,, \tag{3.1.5}$$

where in the second equality we used the Stirling approximation. We now focus on the case $p = 1$, then taking the first derivative of the above expression in relation to 3.1.4 we obtain

$$I_1 = \frac{1}{2}\left[(\psi^{(0)}(N) + \gamma_E)^2 + \zeta(2) - \psi^{(1)}(N)\right]\,, \tag{3.1.6}$$

where γ_E is Euler-Mascheroni constant, $\zeta(n)$ denotes the Riemann zeta function evaluated at n, and $\psi^{(n)}$ is the $n-$th polygamma function, that is defined as follows

$$\psi^{(n)}(x) = \frac{d^{n+1}}{dx^{n+1}} \ln\Gamma(x)\,. \tag{3.1.7}$$

In the large $N-$limit, it can be shown that

$$I_1 = \frac{1}{2}\ln^2(Ne^{\gamma_E}) + \frac{\zeta(2)}{2} + \mathcal{O}\left(\frac{1}{N}\right)\,. \tag{3.1.8}$$

If we repeat the procedure for $p = 0$, we obtain

$$I_0 = -\ln(Ne^{\gamma_E}) + \mathcal{O}\left(\frac{1}{N}\right)\,. \tag{3.1.9}$$

In general, at order $p = n$, we obtain a term proportional to $\ln^{p+1}(Ne^{\gamma_E})$. This provides the leading logarithmic conversion valid for $N \to \infty$,

$$\left[\frac{\ln^p(1-z)}{(1-z)}\right]_+ \Leftrightarrow \ln^{p+1}\left(Ne^{\gamma_E}\right)\,. \tag{3.1.10}$$

In this thesis, in order to obtain the Mellin transformation of the partonic coefficient function we use the correspondence between the Mellin transformation and harmonic sums [27, 28]. Specifically, the analytical continuation of harmonic sums in Mellin space can be exploited to express the Mellin transformation of distributions in terms of harmonic sums. Through this approach, the Mellin transformation of the plus distribution can be rewritten as a linear combination of harmonic sums. Consequently, the behavior in Mellin space for large N can be easily determined by extracting the leading coefficient in the expansion of the harmonic sums. For the definition and a complete overview about the harmonic sums one can see [28].

We now focus on the SIDIS case, which has two scaling variables $\hat{x}$ and $\hat{z}$, then we can have logarithms terms written in terms of either $\hat{x}$ or $\hat{z}$. In particular, at the kth order in perturbation theory, we have the following contributions [29]

- The *delta* contributions, $\alpha_s^k \delta(1-\hat{x})\left[\frac{\ln^n(1-\hat{z})}{(1-\hat{z})}\right]_+$ and $\alpha_s^k \delta(1-\hat{z})\left[\frac{\ln^m(1-\hat{x})}{(1-\hat{x})}\right]_+$ with $n \leq 2k-1$ and $m \leq 2k-1$.
- The *mixed* contributions, $\alpha_s^k \left[\frac{\ln^m(1-\hat{x})}{(1-\hat{x})}\right]_+ \left[\frac{\ln^n(1-\hat{z})}{(1-\hat{z})}\right]_+$ with $m + n \leq 2k-2$

For the SIDIS case, from the above considerations, in the double-soft limit case it is clear that summing the mixed and delta contributions in Mellin space yields contributions of the form

$$\alpha_s^k \mathcal{L}^m \quad m \leq 2k\,, \tag{3.1.11}$$

where

$$\mathcal{L} := \frac{1}{2}(\ln \bar{N} + \ln \bar{M}) \tag{3.1.12}$$

$$\bar{N} = N e^{\gamma_E} \quad \bar{M} = M e^{\gamma_E}\,. \tag{3.1.13}$$

For convenience we include the factor e^{γ_E} which appears in the limit of the logs Mellin transformation in the definition of the Mellin variable. In this manner we simplify a lot the calculations.

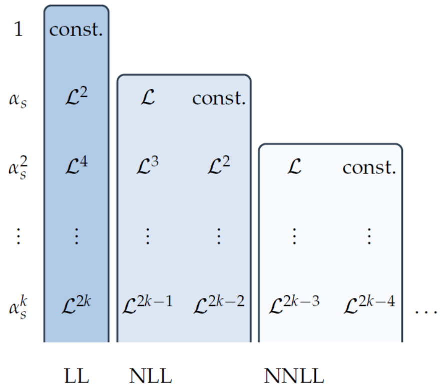


Figure 3.1: All-order threshold logarithmic structure [30]

From Eq. 3.1.11 it is easy to understand that the enhanced logarithms which appear to all orders in perturbation thepry follows a peculiar hierarchy which is displayed in Fig.3.1. Specifically, the row denotes the fixed order calculation while the columns denotes how the logarithms that are summed through the resummation formula, as it is proved in the next chapter. Towers of logs are denoted by the notation LL (leading logarithm), NLL (next-to-leading logarithm), NNLL and so on; where the LL tower contains the most significant corrections.

In conclusion, we express in Mellin space the soft scales on which the partonic cross section depends in the threshold limit. We associate to the $\hat{x}$ scaling the $\bar{N}$ mellin variable, while to $\hat{z}$ the $\bar{M}$ variable.

Therefore, from above considerations for the single-soft limits we obtain that the scales in Eq. 2.2.37 in Mellin space becomes

$$\Lambda_{\rm ss} = \begin{cases} \frac{Q^2}{\bar{N}} & \bar{N} \to \infty, \bar{M} \text{ fixed} \\ \frac{Q^2}{\bar{M}} & \bar{M} \to \infty, \bar{N} \text{ fixed}\,, \end{cases} \tag{3.1.14}$$

Whereas, for the double-soft case, using an approach similar to the one applied above to establish the correspondence between a single-soft scale and its Mellin-space representation, one can, with extra work, show—as done in the appendix of [3]—that the double-soft scale in Eq. 2.2.30 transforms in Mellin space as

$$\Lambda_{\rm ds} = \frac{Q^2}{\bar{N}\bar{M}} \quad \bar{N} \to \infty, \bar{M} \to \infty\,. \tag{3.1.15}$$

In particular, in the appendix of [3] is showed that in the double-soft limit one can use as resummation formula of the DY at fixed rapidity, and so also for the SIDIS process, the same resummation formula of the inclusive case. From [25] we know that the soft scale of the inclusive case in Mellin space is $Q^2/\bar{N}^2$, therefore

$$\tilde{C}^{\text{res}}_{\text{SIDIS}}(\bar{N},\bar{M}) = \tilde{C}^{\text{res}}_{\text{DY},\hat{y}}(\bar{N},\bar{M}) = \tilde{C}^{\text{res}}_{\text{DY}}(\sqrt{\bar{N}\bar{M}})\,, \tag{3.1.16}$$

where $\tilde{C}^{\text{res}}$ is the resummed coefficient function. The aim of the next chapter is to find an explicit expression for $\tilde{C}^{\text{res}}$.

3.1.1 Interpretation of the single-soft resummation

Before proceeding further, it is important to understand the meaning of the single-soft resummation. In Sec. 3.1, we discussed which terms the resummation formula sums in the double-soft limit; however, we still need to interpret the terms summed in the single-soft regime.

Firstly, we note that the contributions from enhanced logarithms to the fixed-order corrections of the coefficient function in both soft limits arise from the distributional terms, i.e., those involving either δ- or $+$-distributions. In fact, all other terms are suppressed, as they are proportional to at least the first power of either $1/(NM)$, $\ln(\bar{N})/(NM)$, or $\ln(\bar{M})/(NM)$, all of which approach zero in both the double-soft and single-soft limits. However, in the single-soft case, one of the two Mellin variables is kept constant, which allows some contributions of the type either $1/N$ or $1/M$ to survive compared to the double-soft limit.

Hence, we conclude that, compared to the double-soft case, the single-soft resummation formula also resums the next-to-leading power (NLP) corrections, which take the form of either $\ln(\bar{N})^k/M^j$ or $\ln(\bar{M})^k/N^j$ with $k \geq 0$, $j > 0$, but never terms like $\ln(N)^k/N^j$ or $\ln(M)^k/M^j$.

3.2 Single-scale resummation

In this section, we derive the resummation formula for the partonic coefficient function in the case of single-scale dependence in the soft limit. As shown in the previous chapter, SIDIS in both the double-soft and single-soft limits serves as an example of single-scale dependence. The following derivation is based on [25] and [31]. The key difference between the two works lies in the further assumption of full factorization of the soft singularities. The goal is to obtain a resummation formula that is valid to all orders. To achieve this, we use a renormalization group argument.

We note that, in this section, in contrast to the notation introduced in Sec. 2.1.4 we avoid to use the Tilde symbol over quantities expressed in Mellin space to keep the notation as simple as possible. The correct notation will be restored in the next section.

We express the SIDIS hadron cross section in double-Mellin space, as shown in Eq.2.1.50 but with the scale dependences restored:

$$\sigma(N,M,Q^2) = \sum_{i,j} f_i(N,\mu_F^2) D_j^{H_2}(M,\mu_F^2)\hat{\sigma}_{ij}\left(N,M,Q^2,\alpha_s(\mu_R^2)\right)\,, \tag{3.2.1}$$

here, μ_R and μ_F are renormalization and factorization scales. Because of the arbitrariness of the two varibles μ_R, μ_F we can choose them to be equal, so $\mu_R = \mu_F = \mu$, therefore:

$$\sigma(N,M,Q^2) = \sum_{i,j} f_i(N,\mu^2) D_j^{H_2}(M,\mu^2)\hat{\sigma}_{ij}\left(N,M,Q^2,\alpha_s(\mu^2)\right)\,. \tag{3.2.2}$$

To obtain the single-scale resummation formula, we use a cross-section that depends on only one Mellin moment, N. By the end of the chapter, we recover the SIDIS case, where both the double-soft and single-soft limits depend on the two Mellin variables, $\bar{N}$ and $\bar{M}$. Finally, we remark that the hadronic cross section cannot depend on an arbitrary scale such as μ.

We note that resummation is expressed for a coefficient function. The coefficient function is defined factoring out of the cross-section the Born-level expression

$$\hat{\sigma}_{ij}\left(N, Q^2, \alpha_s(Q^2)\right) = C_{ij}(N, Q^2/\mu^2, \alpha_s(\mu^2))\sigma^0_{ij}\left(Q^2\right) . \tag{3.2.3}$$

Furthermore, in the soft limit, only diagonal partonic channels are unsuppressed, resummation can be performed independently in the quark singlet and gluon channel and we will consequently suppress the parton indices i, j.

The argument is based on the observation that, due to collinear factorization, the physical anomalous dimension

$$\gamma(N, \alpha_s(Q^2)) = \frac{d}{d\ln Q^2}\ln C\left(N, \alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) , \tag{3.2.4}$$

is renormalization-group invariant and finite when expressed in terms of the renormalized coupling $\alpha_s(\mu^2)$.

We now consider the perturbative expansion of the bare coefficient function in powers of the bare coupling constant, which in Mellin-space is given by:

$$C^b(N, Q^2, \alpha_s^b, \epsilon) = \sum_{n=0}^{\infty}(\alpha_s^b)^n C_n^b(N, Q^2, \epsilon) , \tag{3.2.5}$$

where we adopted the dimensional regularization with $d = 4-2\epsilon$ space-time dimensions and with b means bare. We now recall that the renormalized coefficient function C undergoes multiplicative renormalization. This means that all divergences can be removed from the bare coefficient function $C^b(N, Q^2, \alpha_s^b, \epsilon)$ (Eq.3.2.5) by defining a renormalized coupling constant $\alpha_s(\mu^2)$ according to the implicit equation

$$\alpha_s^b(\mu^2, \alpha_s(\mu^2), \epsilon) = \mu^{2\epsilon}\alpha_s(\mu^2)Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon) , \tag{3.2.6}$$

and a renormalized coefficient function according to

$$C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = Z^{(C)}(N, \alpha_s(\mu^2), \epsilon)C^b(N, Q^2, \alpha_s^b, \epsilon) . \tag{3.2.7}$$

Where $Z^{(\alpha_s)}$ and $Z^{(C)}$ are computable in perturbation theory and have multiple poles at $\epsilon = 0$.

Moreover, thanks to dimensional analysis, we find that the dependence of C^b on Q^2 and α_s^b is only through the dimensionless combination $Q^{-2\epsilon}\alpha_s^b$, namely

$$C^b(N, Q^2, \alpha_s^b, \epsilon) = C^b(N, Q^{-2\epsilon}\alpha_s^b, \epsilon) . \tag{3.2.8}$$

Therefore, by the independence of $Z^{(C)}$ on Q^2, and using Eq.3.2.8 the anomalous dimension γ can be rewritten as:

$$\gamma(N, \alpha_s(Q^2)) = -\epsilon\alpha_s^b\frac{d}{d\alpha_s^b}\ln C^b(N, Q^2, \alpha_s^b, \epsilon) . \tag{3.2.9}$$

Note that for a single-scale process in the soft limit the dimensional dependence of the phase space, therefore of the coefficient function, is through a fixed combination of the scale and the scaling variable

$$\Lambda_a(x, \lambda^2) = \lambda^2(1-x)^a \quad a \in \mathbb{N} , \tag{3.2.10}$$

where, for example, from [25] $a = 1$ for DIS and $a = 2$ for DY, otherwise for the SIDIS one can seen the Eqs. 2.2.37,2.2.30. This implies that Mellin-space coefficient function only depends on N through the dimensional variable

$$\bar{\Lambda}_a(N, \lambda^2) = \frac{\lambda^2}{N^a} \quad a \in \mathbb{N} . \tag{3.2.11}$$

We now consider the further assumption of full factorization of the soft scales singularities. This in turn implies that the bare coefficient function $C^{(0)}$ in Mellin space factorizes in the product of two bare coefficient functions

$$C^b(N, Q^2, \alpha_s^b, \epsilon) = C^{(b,c)}(Q^2, \alpha_s^b, \epsilon)C^{(b,l)}(\bar{\Lambda}_a(N, Q^2), \alpha_s^b, \epsilon)\,, \tag{3.2.12}$$

where $C^{(b,c)}$ collects virtual contributions, that have Born kinematics (hence, $C^{(b,c)}$ does not depend on N), and $C^{(b,l)}$ collects contributions due to real emission. The two bare coefficients function can be written in power series as

$$C^{(b,c)}(Q^2, \alpha_s^b, \epsilon) = \sum_{n=0}^{\infty} \left(\alpha_s^b\right)^n Q^{-2\epsilon n} C_n^{(b,c)}(\epsilon)\ , \tag{3.2.13}$$

$$C^{(b,l)}(\bar{\Lambda}_a(N, Q^2), \alpha_s^b, \epsilon) = \sum_{n=0}^{\infty} \left(\alpha_s^b\right)^n \bar{\Lambda}_a^{-2\epsilon n} C_n^{(b,l)}(\epsilon)\,. \tag{3.2.14}$$

Hence, we can rewrite the anomalous dimension as

$$\gamma(Q^2, \alpha_s(Q^2)) = \gamma^c\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) + \gamma^l\left(\frac{\bar{\Lambda}_a(N, Q^2)}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\ , \tag{3.2.15}$$

$$\gamma^c\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) := -\epsilon\frac{d\ln C^{(0,c)}}{d\ln \alpha_s^b}(Q^2, \alpha_s^b, \epsilon)\,, \tag{3.2.16}$$

$$\gamma^l\left(\frac{\bar{\Lambda}_a(N, Q^2)}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) := -\epsilon\frac{d\ln C^{(0,l)}}{d\ln \alpha_s^b}(\bar{\Lambda}_a(N, Q^2), \alpha_s^b, \epsilon)\,. \tag{3.2.17}$$

It is important to stress that γ^c and γ^l are not individually finite for $\epsilon \to 0$, and thus depend a priori on the scale μ. However, their sum, namely the physical anomalous dimension γ is finite and renormalization group invariant. Hence deriving both sides of Eq.3.2.15 with respect to μ^2 we obtain

$$\lim_{\epsilon\to 0} \frac{d\gamma^c}{d\ln \mu^2}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = -g(\alpha_s(\mu^2))\,, \tag{3.2.18}$$

$$\lim_{\epsilon\to 0} \frac{d\gamma^l}{d\ln \mu^2}\left(\frac{\bar{\Lambda}_a(N, Q^2)}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = g(\alpha_s(\mu^2))\,. \tag{3.2.19}$$

Here, $g(\alpha(\mu^2))$ is finite power series of the coupling constant $\alpha_s(\mu^2)$. Therefore, by solving RGE for both γ^l and γ^c and by adding the solutions, we obtain the following resummation formula for the anomalous dimension:

$$\gamma(N, \alpha_s(Q^2)) = \tilde{g}_0(\alpha_s(Q^2)) + \int_{Q^2}^{\bar{\Lambda}_a^2(N,Q^2)} \frac{dk^2}{k^2} g(\alpha_s(k^2)) \tag{3.2.20}$$

where g_0 is an analytic function of its argument.

Therefore, using Eq. 3.2.20 for the physical anomalous dimensions that emerge from the RG argument, we obtain a resummed expression for the coefficient function of the form

$$C^{\rm res}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = g_0\left(\frac{Q^2}{\mu^2}, \alpha_s(Q^2)\right) \exp\left\{\int_{\mu^2}^{Q^2} \frac{dk^2}{k^2} \int_{k^2}^{\frac{k^2}{N^a}} \frac{d\lambda^2}{\lambda^2} g(\alpha_s(\lambda^2))\right\}, \tag{3.2.21}$$

where

$$g_0(\alpha_s) = 1 + \sum_{i=1}^{\infty} g_0^{(i)} \left(\frac{\alpha_s}{\pi}\right)^i\ , \tag{3.2.22}$$

$$g(\alpha_s) = \sum_{i=1}^{\infty} g_i \left(\frac{\alpha_s}{\pi}\right)^i\,. \tag{3.2.23}$$

We now want to rewrite the resummation formula in an equivalent form, but as we show in the next sections, that it is more convenient for our aims.

We introduce the standard anomalous dimension γ_{AP}, defined by

$$\mu^2 \frac{dF(N,\mu^2)}{d\mu^2} = \gamma_{AP}(N, \alpha_s(\mu^2))F(N,\mu^2)\,, \tag{3.2.24}$$

Where F, in the inclusive case, represents the PDF in DIS or the luminosity in DY, where by luminosity we mean the product of the two PDFs of the incoming partons. Obviously, in non-inclusive cases such as DY and SIDIS, F is a function of both Mellin variables N and M. For instance, in the SIDIS case, it corresponds to the product of the PDF and the FF.

Therefore, the anomalous dimension can be rewritten as

$$\begin{aligned}\gamma(N, \alpha_s(Q^2)) &= \frac{d}{d\ln Q^2} \ln C\left(N, \alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) \\ &= \gamma_{AP}(N, \alpha_s(Q^2)) + \frac{d}{d\ln Q^2} \ln C\left(N, \alpha_s(Q^2), 1\right)\end{aligned} \tag{3.2.25}$$

where, in the second equation we used Eq. (3.2.1) with $Q^2 = \mu^2$.

From the last equation, it follows that the physical and standard anomalous dimensions coincide at leading order in α_s hence at NLO in perturbation theory, but differ beyond leading order. Then, we now note that

$$\begin{aligned}&\int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma(N, \alpha_s(k^2)) = \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma_{AP}(N, \alpha_s(k^2)) \\ &+ \ln C\left(N, \alpha_s(Q^2), 1\right) - \ln C\left(N, \alpha_s(Q_0^2), 1\right)\end{aligned} \tag{3.2.26}$$

It is thus natural to separate in Eq. 3.2.21 the contribution from the standard anomalous dimension and that from the coefficient function. To achieve this, we perform the change of variable $\lambda^2 = k^2/n$ and interchange the integration over k^2 and dn in Eq. 3.2.21:

$$C^{\text{res}}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = g_0\left(\alpha_s(Q^2), \frac{Q^2}{\mu^2}\right) \exp\left\{-\int_1^{N^a} \frac{dn}{n} \int_{n\mu^2}^{Q^2} \frac{dk^2}{k^2} g(\alpha_s(k^2/n))\right\}, \tag{3.2.27}$$

hence, from Eq. 3.2.26 we can rewrite the resummation formula as

$$\begin{aligned}&C^{\text{res}}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = g_0\left(\alpha_s(Q^2), \frac{Q^2}{\mu^2}\right) \times \\ &\exp\left\{\int_1^{N^a} \frac{dn}{n}\left[\int_{Q^2}^{n\mu^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n)) - D(\alpha_s(Q^2/n))\right]\right\},\end{aligned} \tag{3.2.28}$$

where again A, D are power series in α_s. Specifically, A is the contribution given by the standard anomalous dimension while D is the extra terms which provide the difference between γ and γ_{AP}. Consequently, we have:

$$A(\alpha_s) = \sum_{i=1}^{\infty} A^{(i)} \left(\frac{\alpha_s}{\pi}\right)^i, \tag{3.2.29}$$

$$D(\alpha_s) = \sum_{i=i}^{\infty} D^{(i)} \left(\frac{\alpha_s}{\pi}\right)^i. \tag{3.2.30}$$

Including the first $k+1$ terms in the perturbative expansion of A and g_0, and the first k terms in the perturbative expansion of D, leads to resumattion with N^kLL accuracy.

Finally, we can note that for large x it can be proved [32] that only the diagonal splitting functions have an enhancement behaviour, and it is given by:

$$P_{ii}(x) \simeq \frac{A_i}{(1-x)_+} + B_i\delta(1-x)\,. \tag{3.2.31}$$

Thus, from DGLAP equations (Sec. 1.5) A (sometimes called $\Gamma^{\text{cusp}}(\alpha_s)$) is given by

$$\gamma_{AP}(N,\alpha_s) = -\int_0^1 \mathrm{d}x\, x^{N-1}P(x) \tag{3.2.32}$$

$$\lim_{N\to\infty}\gamma_{AP}(N,\alpha_s) = \Gamma^{(\text{cusp})}(\alpha_s)\ln\bar{N} + \text{const.}, \tag{3.2.33}$$

$$\Gamma^{(\text{cusp})}(\alpha_s) = A^{(1)}\frac{\alpha_s}{\pi} + A^{(2)}\left(\frac{\alpha_s}{\pi}\right)^2 + \dots \tag{3.2.34}$$

where P is the splitting function.

As a final step, it is useful to rewrite the resummation formula reported in Eq. 3.2.28 with one less integration. So, taking the exponential

$$\begin{aligned}
&\int_1^{N^a}\frac{\mathrm{d}n}{n}\left[\int_{Q^2}^{n\mu^2}\frac{\mathrm{d}k^2}{k^2}A\left(\alpha_s\left(\frac{k^2}{n}\right)\right) - D\left(\alpha_s\left(\frac{Q^2}{n}\right)\right)\right] = \\
&\int_1^{N^a}\frac{\mathrm{d}n}{n}\left[\int_{Q^2}^{nQ^2}\frac{\mathrm{d}k^2}{k^2}A\left(\alpha_s\left(\frac{k^2}{n}\right)\right) + \int_{nQ^2}^{n\mu^2}\frac{\mathrm{d}k^2}{k^2}A\left(\alpha_s\left(\frac{k^2}{n}\right)\right) - D\left(\alpha_s\left(\frac{Q^2}{n}\right)\right)\right] = \\
&\int_1^{N^a}\frac{\mathrm{d}n}{n}\int_{\frac{Q^2}{n}}^{Q^2}\frac{\mathrm{d}q^2}{q^2}A(\alpha_s(q^2)) + \int_1^{N^a}\frac{\mathrm{d}n}{n}\int_{Q^2}^{\mu^2}\frac{\mathrm{d}q^2}{q^2}A(\alpha_s(q^2)) - \int_{\frac{Q^2}{N^a}}^{Q^2}\frac{\mathrm{d}q^2}{q^2}D(\alpha_s(q^2)) = \\
&\int_{\frac{Q^2}{N^a}}^{Q^2}\frac{\mathrm{d}q^2}{q^2}A(\alpha_s(q^2))\int_{\frac{Q^2}{q^2}}^{N^a}\frac{\mathrm{d}n}{n} + \ln N^a\int_{Q^2}^{\mu^2}\frac{\mathrm{d}q^2}{q^2}A(\alpha_s(q^2)) - \int_{\frac{Q^2}{N^a}}^{Q^2}\frac{\mathrm{d}q^2}{q^2}D(\alpha_s(q^2)) = \\
&\int_{\frac{Q^2}{N^a}}^{Q^2}\frac{\mathrm{d}q^2}{q^2}\left[A(\alpha_s(q^2))\ln\frac{N^a q^2}{Q^2} - D(\alpha_s(q^2))\right] + \ln N^a\int_{Q^2}^{\mu^2}\frac{\mathrm{d}q^2}{q^2}A(\alpha_s(q^2))\,,
\end{aligned} \tag{3.2.35}$$

where in third step we made the change of variables $q^2 = k^2/n$ for the first two integral, while for the third $q^2 = Q^2/n$. Therefore, Eq. 3.2.28 can be written as

$$\begin{aligned}
C^{\text{res}}\left(N,\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right) = g_0\left(\alpha_s(Q^2),\frac{Q^2}{\mu^2}\right)\times \\
\exp\left\{\int_{\frac{Q^2}{N^a}}^{Q^2}\frac{\mathrm{d}q^2}{q^2}\left[A(\alpha_s(q^2))\ln\frac{N^a q^2}{Q^2} - D(\alpha_s(q^2))\right]\right. \\
\left. + \ln N^a\int_{Q^2}^{\mu^2}\frac{\mathrm{d}q^2}{q^2}A(\alpha_s(q^2))\right\}\,.
\end{aligned} \tag{3.2.36}$$

3.2.1 Matching procedure

From Eqs. 3.2.36 and 3.2.28, we see that the resummation formula exponentiates the logarithms, summing them to all orders. What remains to be determined are the coefficients of the power series in the exponent, in our notation (A_i and D_i), and the constant factors ($g_0^{(i)}$). To reach this aim, we can solve the integrals in the exponential of our resummation formula, then we can expand it to a fixed order in perturbation theory, for instance at order α_s^n.

This computation should yield the fixed-order result (in Mellin space), which must coincide with the fixed-order calculation of the partonic cross section in the same space in the threshold limit. By comparing the two expressions, we can extract the coefficients of the power series, which can then be substituted back into the all-order resummation formula. From Fig. 3.1 it is clear which towers of logarithms we are able to determine through the coefficient obtained by the comparison with the $n-$th fixed order. In particular, from the N^{n}LO fixed order we predict the towers of logs up to N^{n}LL.

In the next chapter, we determine the coefficients up to NNLL for the resummation of the SIDIS process.

Now that we have determined how to obtain the resummation formulas, it is important to note that these formulas are valid only in the threshold region. Outside of this kinematic regime, we must rely on the fixed-order calculation. To combine both results, we must subtract the fixed-order expansion of the resummation formula up to the highest order of the fixed-order calculation, ensuring that these terms do not appear twice, as that would lead to a non-physical result.

3.2.2 SIDIS resummation formulas

In this thesis, for simplicity we work with $Q^2 = \mu_F^2 = \mu_R^2$, because the dependence by the arbitrary scales μ_R^2 and μ_F^2 can be restored at the end of the calculations through the RGE and DGLAP equations. Therefore, from Eq. 3.1.16 and from the formula in Eq.3.2.36 obtained for the resummation formulas for processes that have a single-scale dependence in the soft-limit, we obtain,

double-soft limit resummation formula:

$$\tilde{C}^{\text{res}}\left(N, M, \alpha_s(Q^2)\right) = g_0\left(\alpha_s(Q^2)\right) \times \\ \exp\left\{\int_{\frac{Q^2}{\bar{N}\bar{M}}}^{Q^2} \frac{\mathrm{d}k^2}{k^2}\left[A(\alpha_s(k^2))\ln\frac{\bar{N}\bar{M}k^2}{Q^2} - D(\alpha_s(k^2))\right]\right\}, \tag{3.2.37}$$

or equivalently using Eq. 3.2.28

$$\tilde{C}^{\text{res}}\left(N, M, \alpha_s(Q^2)\right) = g_0\left(\alpha_s(Q^2)\right) \times \\ \exp\left\{\int_1^{\bar{N}\bar{M}} \frac{dn}{n}\left[\int_{Q^2}^{nQ^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n)) - D(\alpha_s(Q^2/n))\right]\right\}. \tag{3.2.38}$$

On the other hand, for the single-soft limits, the behaviour of the resummation formula is somewhat different. In both single-soft limits, we have a single-scale dependence, as shown in Sec. 2.2.1. However, now one of the two Mellin variables is constant, meaning that the coefficients of the resummation formula must depend on this Mellin variable. Consequently, from the single-soft scales in Eq. 3.1.14, we obtain,

$\hat{x}$-single-soft limit resummation formula:

$$\tilde{C}^{\text{res}}\left(N, M, \alpha_s(Q^2)\right) = g_0\left(M, \alpha_s(Q^2)\right) \times \\ \exp\left\{\int_{\frac{Q^2}{\bar{N}}}^{Q^2} \frac{\mathrm{d}k^2}{k^2}\left[A(M, \alpha_s(k^2))\ln\frac{\bar{N}k^2}{Q^2} - D(M, \alpha_s(k^2))\right]\right\}, \tag{3.2.39}$$

or equivalently

$$\tilde{C}^{\mathrm{res}}\left(N, M, \alpha_s(Q^2)\right) = g_0\left(M, \alpha_s(Q^2)\right) \times$$
$$\exp\left\{\int_1^{\bar{N}} \frac{dn}{n}\left[\int_{Q^2}^{nQ^2} \frac{dk^2}{k^2} A(M, \alpha_s(k^2/n)) - D(M, \alpha_s(Q^2/n))\right]\right\}, \tag{3.2.40}$$

$\hat{z}$-single-soft limit resummation formula:

$$\tilde{C}^{\mathrm{res}}\left(N, M, \alpha_s(Q^2)\right) = g_0\left(N, \alpha_s(Q^2)\right) \times$$
$$\exp\left\{\int_{\frac{Q^2}{\bar{M}}}^{Q^2} \frac{\mathrm{d}k^2}{k^2}\left[A(N, \alpha_s(k^2)) \ln \frac{\bar{M}k^2}{Q^2} - D(N, \alpha_s(k^2))\right]\right\}, \tag{3.2.41}$$

or equivalently

$$\tilde{C}^{\mathrm{res}}\left(N, M, \alpha_s(Q^2)\right) = g_0\left(N, \alpha_s(Q^2)\right) \times$$
$$\exp\left\{\int_1^{\bar{M}} \frac{dn}{n}\left[\int_{Q^2}^{nQ^2} \frac{dk^2}{k^2} A(N, \alpha_s(k^2/n)) - D(N, \alpha_s(Q^2/n))\right]\right\}. \tag{3.2.42}$$

Chapter 4

NNLL SIDIS resummation

In this chapter, we derive the resummation formula coefficients at NNLL accuracy for the qq channel of the SIDIS process in both the double-soft and single-soft limits, using the matching procedure. This channel contains, in fact, the most significant logarithmic corrections.

In particular, in Sec. 4.2 we report the theoretical predictions for the DY at fixed rapidity provided by [3] and from then we establish the theoretical predictions at NNLL for the SIDIS case. After that, we verify explicitly these predictions for SIDIS case, providing for the first time the single-soft resummation of this process.

4.1 NS transverse coefficient function up to NNLO

For the matching procedure Sec. 3.2.1, we need a fixed-order result at NNLO. To this aim, we use the SIDIS coefficient functions C_{qq} recently obtained in [26] at NNLO. We note that, more recently, in [33] they obtained the polarized case and also the same results of [26] for the unpolarized case at NNLO. As it shown in Secs. 3.1.1 and 3.1 the double- single-soft limits of the coefficient function are provided by its delta and plus-distribution contributions.

In this section, we report some properties of the distributional behaviour of the coefficient function C_{qq}. As shown in Sec. 2.1, all coefficient functions are split into two contributions, representing the perturbative contributions to the transverse structure function ($\mathcal{F}_T$), namely C^T_{qq}, and to the longitudinal structure function ($\mathcal{F}_L$), namely C^L_{qq}. The aforementioned work provides both C^T_{qq} and C^L_{qq}.

First, we note that C^L_{qq} does not contain distributional contributions; therefore, we focus only on the study of C^T_{qq}. Furthermore, as shown in Sec. 1.5.1, beyond NLO, C^T_{qq} can be split into a NS and a PS contribution. Again, the PS contribution does not contain any distributional terms.

Therefore, the resummation problem concerns only the non-singlet contribution C^T_{qq} up to NNLO. Hence, the distributional part of the coefficient function can be written as a perturbative expansion in α_s as follows

$$C^{T,d}_{qq} = C^{(0)}_{qq} + \left(\frac{\alpha_s}{\pi}\right) C^{T,d,(1)}_{qq} + \left(\frac{\alpha_s}{\pi}\right)^2 C^{T,d,\mathrm{NS},(2)}_{qq} + \mathcal{O}(\alpha_s^3)\,, \tag{4.1.1}$$

where $C^{T,d}$ represents the distributional part of the coefficient function. Furthermore at the LO from Callan-Gross relation the longitudinal contribution is not present, in particular

$$C^{(0)}_{qq}(\hat{x},\hat{z}) = \delta(1-\hat{x})\delta(1-\hat{z})\,, \tag{4.1.2}$$

that in Mellin space becomes

$$\tilde{C}^{(0)}_{qq}(N,M) = 1\,. \tag{4.1.3}$$

Finally, taking the mellin-transformation of Eq. 4.1.1, in either single- or double-soft limit $\tilde{C}^T_{qq}$ could be expressed as:

$$\tilde{C}^{T,d}_{qq}(N,M) = 1 + \left(\frac{\alpha_s}{\pi}\right)\tilde{C}^{T,d,(1)}_{qq}(N,M) + \left(\frac{\alpha_s}{\pi}\right)^2 \tilde{C}^{T,d,\mathrm{NS},(2)}_{qq}(N,M) + \mathcal{O}(\alpha_s^3)\,, \tag{4.1.4}$$

$$\tilde{C}^T_{qq}(N,M) \simeq \tilde{C}^{T,d}_{qq}(N,M) \quad \text{when either } N,\, M \text{ or both approach to } \infty. \tag{4.1.5}$$

where N, M are the mellin moments of $\hat{x}, \hat{z}$ respectively, hence, as $\hat{x} \to 1$ then $N \to \infty$ and as $\hat{z} \to 1$ then $M \to \infty$. We remember that $\tilde{C}^T$ is the Mellin transformation of the coefficient function C^T, then $\tilde{C}^{T,d}$ is the Mellin transformation of the distributional part of the coefficient function. Finally, as it shown in the next section, we observe that the single-soft cases the coefficient function needs a further modification.

In order to provide the Mellin transformation of the distributional part of the coefficient function we extracted them from the contributions to the coefficient function at NLO $C^{T,(1)}_{qq}$ and at NNLO $C^{T,(2)}_{qq}$ provided in [26]. In this manner, we have also showed that C^L and C^{PS} do not contain any distributional contribution.

Therefore, we derived the Mellin transformation of the distributional contributions of C^T using the `Mathematica` package `MT` in particular its function `MTMellin` (for the package usage see [34]). Consequently, we proceed to compute their asymptotic behaviour in single- and double-soft limits. As shown in appendix A.2, the Mellin transformation of the plus distributions that appear in the QCD coefficient functions leads to harmonic sums. Hence, to treat the asymptotic behaviour in Mellin space of the coefficients functions $\tilde{C}^{T,(1)}_{qq}$ and $\tilde{C}^{T,(2)}_{qq}$ we used the `Mathematica` package `HarmonicSums`, in particular its function `SExpansion` (for the package usage see [28]).

4.2 Theoretical predictions at NNLL

In this section we provide a theoretical prediction for the SIDIS resummation formula at NNLL in both double- and single-soft limits based on the result provided in [3] for the DY process at fixed rapidity. We anticipate that the goal of the next sections is to verify these predictions, consequently providing for the first time the asymmetric resummation for the SIDIS process. For the DY we follow the notation introduced in Eq. 2.1.21 in Mellin-space we associate to the $\hat{x}_1$ variable the moment N, while to the variable $\hat{x}_2$ the moment M.

As it shown in [3] and [4], and as we verify explicitly in the next chapter, in the double-soft limit at NNLL the resummation formula for both SIDIS and DY at fixed rapidity in Eq. 3.2.37 becomes

$$\begin{aligned}&\tilde{C}^{\mathrm{ds},T}_{qq}\left(N,M,\alpha_s(Q^2)\right) = g_{0,(\mathrm{ds},qq)}\left(\alpha_s(Q^2)\right) \times \\ &\exp\left\{\int_{\frac{Q^2}{\bar{N}\bar{M}}}^{Q^2} \frac{\mathrm{d}k^2}{k^2}\left[A_{qq}(\alpha_s(k^2))\ln\frac{\bar{N}\bar{M}k^2}{Q^2} - D^{(2)}_{\mathrm{ds},qq}\frac{\alpha_s^2(k^2)}{\pi^2}\right]\right\}\,,\end{aligned} \tag{4.2.1}$$

where with the subscript ds we mean double-soft and A is the cusp function, then

$$A_{qq}(\alpha_s) = A^{(1)}_{qq}\frac{\alpha_s}{\pi} + A^{(2)}_{qq}\frac{\alpha_s^2}{\pi} + A^{(3)}_{qq}\frac{\alpha_s^3}{\pi} + \dots\,, \tag{4.2.2}$$

and

$$g_{0,(\mathrm{ds},qq)}(\alpha_s) = 1 + g^{(1)}_{0,(\mathrm{ds},qq)}\frac{\alpha_s}{\pi} + g^{(2)}_{0,(\mathrm{ds},qq)}\left(\frac{\alpha_s}{\pi}\right)^2 + \dots \tag{4.2.3}$$

Therefore, in double-soft limit the D_1 term vanishes. In the next section, we verify up to α_s^2 that the coefficients A_{qq} are the coefficients of the cusp, and we obtain the explicit expression for $D^{(2)}_{qq}$,$g^{(1)}_{0,(\mathrm{ds},qq)}$ and $g^{(2)}_{0,(\mathrm{ds},qq)}$.

The work in [3] also presents a resummation formula for the DY process at fixed rapidity up to NNLL. Specifically, this resummation formula using the soft-scale of the double-soft limit taking one of the two Mellin variables as fixed produce both the double- and single-soft limits. In this manner, one can obtain the double-soft coefficients simply by sending the fixed Mellin variable to infinity.

This result is derived in [3] by translating a previous finding obtained within the framework of Soft-Collinear Effective Theory (SCET) [35] into QCD. However, explaining how SCETs work is beyond the scope of this thesis, but an introduction to the topic can be found in [13]. Therefore, we directly report the translated result. For instance, for the $\hat{x}_1-$single-soft limit we have

$$\tilde{C}_{qq}^{\mathrm{SCET},T}\left(N, M, \alpha_s(Q^2)\right) = \hat{H}\left(\alpha_s(Q^2)\right)\exp\left\{\int_{\frac{Q^2}{\bar{N}\bar{M}}}^{Q^2}\frac{\mathrm{d}k^2}{k^2}\left[\Gamma_{\mathrm{cusp}}(\alpha_s(k^2))\ln\frac{\bar{N}\bar{M}k^2}{Q^2}\right.\right.$$
$$\left.\left.-\hat{P}_{qq}^{(0)}(M)\frac{\alpha_s(k^2)}{\pi} - \left(D_{\mathrm{ds},qq}^{(2)} - \pi\beta_0 F(M) + \hat{P}_{qq}^{(1)}(M)\right)\frac{\alpha_s^2(k^2)}{\pi^2}\right]\right\}, \tag{4.2.4}$$

where, taking the space-like splitting function P_{qq}, we define, with a slight abuse of notation,

$$P_{qq}(M) = \int_0^1 \mathrm{d}\hat{x}_2^{M-1} P_{qq}(\hat{x}_2)\,, \tag{4.2.5}$$

$$P_{qq}(M) = P_{qq}^{(0)}\frac{\alpha_s}{\pi} + P_{qq}^{(1)}\frac{\alpha_s^2}{\pi} + \dots\,, \tag{4.2.6}$$

$$\hat{P}_{qq}^{(i)}(M) = P_{qq}(M) - \lim_{M\to\infty} P_{qq}(M)\,, \tag{4.2.7}$$

whereas the prefactor $\hat{H}$ is given by

$$\hat{H} = 1 + g_{0,(\mathrm{ds},qq)}^{(1)}\frac{\alpha_s}{\pi} + F(M)\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\,. \tag{4.2.8}$$

Thus, it is straightforward to generalize the double-soft result in order to match the SCET result. In fact, we can do it simply writing

$$\tilde{C}_{qq}^{\mathrm{gen},T}\left(N, M, \alpha_s(Q^2)\right) = g_{0,qq}^{\mathrm{gen}}\left(M, \alpha_s(Q^2)\right)\times$$
$$\exp\left\{\int_{\frac{Q^2}{\bar{N}\bar{M}}}^{Q^2}\frac{\mathrm{d}k^2}{k^2}\left[A_{qq}(\alpha_s(k^2))\ln\frac{\bar{N}\bar{M}k^2}{Q^2} - D_{qq}^{\mathrm{gen}}(M, \alpha_s(k^2))\right]\right\}, \tag{4.2.9}$$

where

$$g_{0,qq}^{\mathrm{gen}}(M, \alpha_s) = 1 + g_{0,qq}^{\mathrm{gen},(1)}(M)\frac{\alpha_s}{\pi} + g_{0,qq}^{\mathrm{gen},(2)}(M)\left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}(\alpha_s^3) \tag{4.2.10}$$

$$D_{qq}^{\mathrm{gen}}(M, \alpha_s) = D_{qq}^{\mathrm{gen},(1)}(M)\frac{\alpha_s}{\pi} + D_{qq}^{\mathrm{gen},(2)}(M)\left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}(\alpha_s^3) \tag{4.2.11}$$

and

$$g_{0,(\mathrm{ds},qq)}^{(1)} \to g_{0,qq}^{\mathrm{gen},(1)}(M) \equiv \hat{H}_1 = g_{0,(\mathrm{ds},qq)}^{(1)} + F(M)\,, \tag{4.2.12}$$

$$D_{\mathrm{ds},qq}^{(1)} \to D_{qq}^{\mathrm{gen},(1)}(M) = \hat{P}_{qq}^{(0)}(M)\,, \tag{4.2.13}$$

$$D_{\mathrm{ds},qq}^{(2)} \to D_{qq}^{\mathrm{gen},(2)}(M) = D_{\mathrm{ds},qq}^{(2)} - \pi\beta_0 F(M) + \hat{P}_{qq}^{(1)}(M) \tag{4.2.14}$$

In particular

$$F(M) = \hat{P}_{qq}^{(0)}(M) \ln \bar{M} + \text{other terms}\,, \tag{4.2.15}$$

$$\lim_{M\to\infty} F(M) = 0\,, \tag{4.2.16}$$

$$\lim_{M\to\infty} \hat{P}_{qq}^{(i)}(M) = 0\,, \tag{4.2.17}$$

$$\lim_{M\to\infty} g_{0,qq}^{\text{gen},(2)}(M) = g_{0,(\text{ds},qq)}^{(2)}\,. \tag{4.2.18}$$

the complete definition of $F(M)$ for the DY at fixed rapidity is given in [3]. In the next sections, we provide the complete $F(M)$ function for the SIDIS case.

In DY at fixed rapidity, the $\hat{x}_1$- and $\hat{x}_2$-single-soft cases are identical since both variables are the scaling variable of a PDF. Consequently, the splitting functions appearing in the resummation formula are space-like in both cases. Therefore, changing M with N into the Eq. 4.2.9 we obtain the resummation formula in the $\hat{x}_2-$single-soft limit.

On the contrary, in the SIDIS case, the behavior in the $\hat{x}$- and $\hat{z}$-single-soft limits differs: $\hat{x}$ is the scaling variable of a PDF, while $\hat{z}$ is the scaling variable of a FF. Therefore:

- $\hat{x}-$single-soft limit: we need to use the time-like splitting function $P_{qq}^{T,\text{NS}}(\hat{z})$ in Mellin space;
- $\hat{z}-$single-soft limit: we need to use the space-like splitting function $P_{qq}^{\text{NS}}(\hat{x})$ in Mellin sapce.

It is important to note that in the notation of [26] the NS contributions that into account all the contributes $q \to q$. Therefore, it also considers diagrams of the type $q \to qq\bar{q}$, where there are two quark lines of the same flavour and one tags a q from one line and a q from the other. These terms in double-soft limit vanish, but in the single-soft limit they are still present. In this case, we cannot use the P_{qq} splitting function in order to predict the soft corrections due to these diagrams, because we are considering only the case where we have only one fermion line. So, to verify our theoretical predictions in the single-soft limit we have to subtract them from the coefficient function C_{qq}.

Finally, we observe that in Eq. 4.2.9 we present the resummation formula for the single-soft case of the DY process at fixed rapidity, which, with the above considerations, also applies to the SIDIS case. However, we used the soft-scale of the double-soft limit, namely $Q^2/(\bar{N}\bar{M})$. Thus, in Eq. 4.2.9, the exponential also resums the $\ln \bar{M}$ terms, which are not large. In fact, when we expand the resummation formula, these terms must vanish. It is now clear why we use $\hat{P}$ instead of P: we need to subtract the logarithmic terms that are not large in the single-soft limit. Indeed, as shown in [3], if we use the scale $Q^2/\bar{N}$ instead of $Q^2/\bar{N}\bar{M}$, the resummation formula takes the form of Eq. 4.2.9, but written in terms of the splitting functions without the constant terms and the function $F(M)$ without its logarithmic contributions.

4.3 Resummation formula expansion

To verify the theoretical predictions at NNLL from the previous section, we need to extract the coefficients of the resummation formulas through the matching procedure. This requires comparing the resummation formulas expanded up to NNLO with the soft-limits of the coefficient function, also computed up to NNLO. Here, we provide the expanded expression of the SIDIS resummation formulas obtained in Sec.3.2.2.

We start with the expanded expression for the double-soft limit. We take the resummation formula as in Eq. 3.2.38. Firstly, we rewrite the first integral using RGE equation for the strong coupling as

$$\int_{nQ^2}^{Q^2} \frac{dk^2}{k^2} \alpha_s(k^2/n) = \int_{\alpha_s}^{\alpha_s(Q^2/n)} \frac{d\alpha}{\beta(\alpha)} \alpha \tag{4.3.1}$$

with $\alpha_s = \alpha_s(Q^2)$ and

$$\begin{aligned}\beta(\alpha) &= -\alpha^2 \left(\beta_0 + \beta_1 \alpha + \beta_2 \alpha^2 + \mathcal{O}\left(\alpha^3\right)\right) \\ &= -\alpha^2 b_0 \left(1 + b_1 \alpha + b_2 \alpha^2 + \mathcal{O}\left(\alpha^3\right)\right).\end{aligned} \tag{4.3.2}$$

Where $b_0 = \beta_0$ and for $i \neq 0$ $b_i = \frac{\beta_i}{\beta_0}$. The values of β_k up to $k = 2$ are given in appendix. B.

Whereas the second integrand can be rewritten with the following change of variables

$$\int_1^{\bar{M}\bar{N}} \frac{dn}{n} = -\frac{1}{b_0 \alpha_s} \int_1^{1-\lambda} dl \,, \tag{4.3.3}$$

with

$$l \equiv 1 - b_0 \alpha_s \ln(n) \,, \tag{4.3.4}$$

$$\lambda \equiv b_0 \alpha_s \ln(\bar{N}\bar{M}) \,. \tag{4.3.5}$$

We can apply this change of variable because, by using the RGE, the solutions of $\alpha_s(Q^2/n)$ are functions of α_s and l; see Appendix B for the solutions up to third order. Hence, we can write $\alpha_s(Q^2/n) = \alpha_s(l)$.

At NNLL we need to expand A and g_0 up to order 3 and D up to order 2, hence the resummation formula becomes:

$$\begin{aligned}\tilde{C}_{qq}^{\mathrm{ds},T}\left(N, M, \alpha_s(Q^2)\right) = &\left(1 + g_0^{qq,(1)} \alpha_s(Q^2) + g_0^{qq,(2)} \alpha_s(Q^2)^2\right) \times \\ &\exp\left\{-\frac{1}{b_0 \alpha_s} \int_1^{1-\lambda} dl \int_{\alpha_s}^{\alpha_s(l)} d\alpha \left[\frac{A_{qq}^{(1)}}{\pi b_0} \frac{1}{\alpha + b_1 \alpha^2 + b_2 \alpha^3}\right.\right. \\ &\left. + \frac{A_{qq}^{(2)}}{\pi^2 b_0} \frac{1}{1 + b_1 \alpha + b_2 \alpha^2} + \frac{A_{qq}^{(3)}}{\pi^3 b_0} \frac{\alpha}{1 + b_1 \alpha + b_2 \alpha^2}\right] + \\ &\left. + \frac{1}{b_0 \alpha_s} \int_1^{1-\lambda} dl D_{qq}^{(1)} \frac{\alpha_s(l)}{\pi} + D_{qq}^{(2)} \left(\frac{\alpha_s(l)}{\pi}\right)^2\right\} + \mathcal{O}\left(\alpha_s^3\right) .\end{aligned} \tag{4.3.6}$$

Then, we perform the integrations in the exponential up to order α_s obtaining

$$\begin{aligned}\tilde{C}_{qq}^{\mathrm{ds},T}\left(N, M, \alpha_s(Q^2)\right) = &\left(1 + g_0^{qq,(1)} \alpha_s(Q^2) + g_0^{qq,(2)} \alpha_s(Q^2)^2\right) \times \\ &\exp\left(\frac{1}{\alpha_s} g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda)\right) + \mathcal{O}\left(\alpha_s^3\right) ,\end{aligned} \tag{4.3.7}$$

with

$$g_1(\lambda) = \frac{A_{qq}^{(1)}}{\pi b_0^2}(\lambda + (1-\lambda)\ln(1-\lambda))\,, \tag{4.3.8}$$

$$\begin{aligned} g_2(\lambda) &= \frac{A_{qq}^{(1)} b_1}{2\pi b_0^2}\left(2\lambda + \log^2(1-\lambda) + 2\ln(1-\lambda)\right) - \frac{A_{qq}^{(2)}}{\pi^2 b_0^2}(\lambda + \log(1-\lambda)) \\ &+ \frac{D_{qq}^{(1)}\ln(1-\lambda)}{\pi b_0}\,, \\ g_3(\lambda) &= \frac{A_{qq}^{(1)} b_1^2}{\pi b_0^2(1-\lambda)}\left(\frac{\lambda^2}{2} + \frac{1}{2}\ln^2(1-\lambda) + \lambda\ln(1-\lambda)\right) \\ &+ \frac{A_{qq}^{(1)} b_2}{\pi b_0^2(1-\lambda)}\left(\log(1-\lambda) - \lambda\log(1-\lambda) - \frac{\lambda^2}{2} + \lambda\right) \\ &- \frac{A_{qq}^{(2)} b_1}{\pi^2 b_0^2(1-\lambda)}\left(\frac{\lambda^2}{2} + \lambda + \log(1-\lambda)\right) + \frac{A_{qq}^{(3)}\lambda^2}{2\pi^3 b_0^2(1-\lambda)} \\ &- \frac{D_{qq}^{(2)}\lambda}{\pi^2 b_0(1-\lambda)} + \frac{D_{qq}^{(1)} b_1}{\pi b_0(1-\lambda)}(\lambda + b_1\ln(1-\lambda))\,. \end{aligned} \tag{4.3.9}$$

we note that the functions $g_1(\lambda)$, $g_2(\lambda)$, and $g_3(\lambda)$ are identical to the ones appearing in the exponent of the resummation formula used in [4].

We now expand Eq.(4.3.7) up to order α_s^2, therefore:

$$\begin{aligned} \tilde{C}_{qq}^{\mathrm{ds},T}\left(N, M, \alpha_s(Q^2)\right) \sim& 1 + \frac{\alpha_s}{\pi}\left(2A_1\mathcal{L}^2 - 2D_1\mathcal{L} + g_0^{(1)}\right) \\ &+ \frac{\alpha_s^2}{\pi^2}\left(2A_1^2\mathcal{L}^4 + \frac{4}{3}A_1\pi b_0\mathcal{L}^3 - 4A_1D_1\mathcal{L}^3 + 2A_1g_0^{(1)}\mathcal{L}^2\right. \\ &\left.+2A_2\mathcal{L}^2 - 2\pi b_0 D_1\mathcal{L}^2 + 2D_1^2\mathcal{L}^2 - 2D_1g_0^{(1)}\mathcal{L} - 2D_2\mathcal{L} + g_0^{(2)}\right)\,, \end{aligned} \tag{4.3.10}$$

where for simplicity we omitted the qq index in resummation coefficients and we remember that $\mathcal{L} \equiv \frac{1}{2}(\ln M + \ln N)$. Hence, in order to extract the resummation coefficients in the double-soft limit we compare the Eq. 4.3.10 with the fixed order result.

We now proceed to obtain the single-soft resumattion formula expanded up to α_s^2. If we want to use the scale $Q^2/(\bar{N}\bar{M})$ the expanded expression is simply given by 4.3.10 with $\ln\bar{M}$ and $\ln\bar{N}$ written explicitly. Then, for instance, in the $\hat{x}$-single-soft limit, where, for simplicity, we again omit the qq subscript and the M dependence in the coefficients, we obtain:

$$
\begin{aligned}
\tilde{C}^{\mathrm{res},T}_{qq}\left(N,M,\alpha_s(Q^2)\right) \sim 1 + \frac{\alpha_s}{\pi}\Bigg(&\frac{1}{2}A_1\ln^2\bar{M} - D_1\ln\bar{M} + g_0^{(1)} \\
&+\ln\bar{N}(A_1\ln\bar{M} - D_1) + \frac{A_1\ln^2\bar{N}}{2}\Bigg) \\
&+\frac{\alpha_s^2}{\pi^2}\Bigg\{\frac{A_1^2\ln^4\bar{M}}{8} + \frac{A_1\pi\beta_0\ln^3\bar{M}}{6} - \frac{A_1D_1\ln^3\bar{M}}{2} \\
&+\frac{A_1g_0^{(1)}\ln^2\bar{M}}{2} + \frac{A_2\ln^2\bar{M}}{2} \\
&-\frac{\beta_0\pi D_1\ln^2\bar{M}}{2} + \frac{D_1^2\ln^2\bar{M}}{2} - D_1g_0^{(1)}\ln\bar{M} - D_2\ln\bar{M} + g_0^{(2)} \\
&+\ln\bar{N}\Bigg[\frac{1}{2}A_1^2\ln^3\bar{M} - \frac{3}{2}A_1D_1\ln^2\bar{M} + A_1g_0^{(1)}\ln\bar{M} + A_2 \\
&\ln\bar{M} + D_1^2\ln\bar{M} - D_1g_0^{(1)} - D_2 \\
&+\frac{1}{2}A_1\pi\beta_0\ln^2\bar{M} - \beta_0\pi D_1\ln\bar{M}\Bigg] \\
&+\ln^2\bar{N}\Bigg[\frac{3}{4}A_1^2\ln^2\bar{M} - \frac{3}{2}A_1D_1\ln\bar{M} + \frac{A_1g_0^{(1)}}{2} + \frac{A_2}{2} + \frac{D_1^2}{2} \\
&+\frac{1}{2}A_1\beta_0\pi\ln\bar{M} - \frac{\beta_0\pi D_1}{2}\Bigg] \\
&+\ln^3\bar{N}\left(\frac{1}{2}A_1^2\ln\bar{M} - \frac{A_1D_1}{2} + \frac{A_1\beta_0\pi}{6}\right) + \frac{A_1^2\ln^4\bar{N}}{8}\Bigg\}\,.
\end{aligned} \tag{4.3.11}
$$

Noting that the various coefficients in the above case are functions of the Mellin variable M, then the $\hat{z}$-single-soft limit is analogous; we simply need to exchange N and M.

Finally, if we express the single-soft limit using the correct scale, namely $Q^2/\bar{N}$ for the $\hat{x}$-single-soft limit and $Q^2/\bar{M}$ for the $\hat{z}$-single-soft limit, the procedure is the same. We simply need to make the following re-definition in the calculations above

- $\lambda \equiv b_0\alpha_s\ln\bar{N}$ for the $\hat{x}-$single soft case;
- $\lambda \equiv b_0\alpha_s\ln\bar{M}$ for the $\hat{z}-$single soft case.

Therefore, for $\hat{x}-$single-soft the resummation formula in Eq. 3.2.39 expanded up to α_s^2 becomes

$$
\begin{aligned}
\tilde{C}^{\mathrm{ss},T}_{qq}\left(N,M,\alpha_s(Q^2)\right) &= 1 + \frac{\alpha_s}{\pi}\left(\frac{1}{2}A_1\ln^2\bar{N} - D_1\ln\bar{N} + g_0^{(1)}\right) \\
&+\frac{\alpha_s^2}{\pi^2}\Bigg[g_0^{(2)} + \ln\bar{N}\left(-D_1g_0^{(1)} - D_2\right) \\
&+\ln^2\bar{N}\left(\frac{A_1g_0^{(1)}}{2} + \frac{A_2}{2} - \frac{b_0\pi D_1}{2} + \frac{D_1^2}{2}\right) + \ln^3\bar{N}\left(\frac{A_1\pi b_0}{6} - \frac{A_1D_1}{2}\right) \\
&+\frac{A_1^2\ln^4\bar{N}}{8}\Bigg]\,.
\end{aligned} \tag{4.3.12}
$$

As in the previous case, the $\hat{z}$-single-soft limit is analogous; we simply need to exchange N and M.

4.4 Double-soft limit

In this section we derive the resummation formula at NNLL accuracy for the qq channel of the SIDIS process in the double-soft limit.

This goal has already been achieved in previous works: at NNL accuracy in [36], at NNLL in [29], and more recently also at N^{3}LL in [4]. However, there is a substantial difference between the procedure followed in those works and ours. Because, in the aforementioned works, the resummation coefficients were extracted through a matching procedure based on the conjecture proposed in [37], which suggests that the SIDIS process is the crossed version of Drell-Yan, a statement that we have proven in a more rigorous manner in Sec. 2.2.2. Consequently, according to this correspondence, the SIDIS coefficient functions are derived from the Drell-Yan results, that is from $C_{q\bar{q}}^{\mathrm{DY}}$ we obtain C_{qq}^{SIDIS}. Instead, we directly use the SIDIS coefficient functions up to NNLO, as we mention in the section 4.1. Hence, we have a validity check for both the resummation formalism introduced in the previous chapter, and for the crossed-process correspondence. After this check, we can use the resummation formula obtained by the RGE argument and the crossing correspondence to study new cases of interest for SIDIS process, namely the single-soft cases.

As it shown in the previous section, the integration in the exponential of the resummation formula leads to the same result of [29]. So this check the validity of our resummation formula. Otherwise, one can see that the resummation formula written as in Eq. 3.2.38 is the same resummation formula used in [29] and [4].

We observe that the coefficient functions provided in [26] are expressed in terms of C_A^2, C_A, and N_F, where $C_A = 3$ and N_F represent the number of colors and the number of active flavors in QCD, respectively. Once we obtain the asymptotic expression of the Mellin transform of the coefficient function in the double-soft limit—using the computational tools cited in Sec. 4.1—we rewrite this limit in terms of $C_F = T_R \frac{C_A^2-1}{C_A}$, C_A, and N_F, where $T_R = 1/2$ and $C_F = 4/3$ in QCD. this choice explicitly highlights that the resummation terms As are the coefficients of the perturbative expansion of the cusp function.

Therefore, by performing this transformation and comparing their results with ours, we find full equivalence between the results directly obtained from the SIDIS coefficient functions in our case and those derived by exploiting the crossing symmetry of the Drell-Yan process in [29].

Hence, the final step is the matching between resummation formula and the fixed order result.

At NLO, in the asymptotic limit we obtain

$$\tilde{C}_{qq}^{T,(1)} = C_F \left\{ 2\mathcal{L}^2 + \frac{\pi^2}{6} - 4 \right\} + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(\frac{1}{M}\right) . \tag{4.4.1}$$

While at NNLO, in the asymptotic limit we obtain

$$\begin{aligned}
\tilde{C}_{qq}^{T,\mathrm{NS},(2)}(N,M) \;=\;& 2C_F^2\mathcal{L}^4 + 4C_F\mathcal{L}^3\left(\frac{\pi}{3}b_0\right) \\
&+ C_F\mathcal{L}^2\left[C_F\left(-8+\frac{\pi^2}{3}\right)+\left(\frac{67}{18}-\frac{\pi^2}{6}\right)C_A-\frac{5}{9}N_f\right] \\
&+ C_F\mathcal{L}\left[\left(\frac{101}{27}-\frac{7}{2}\zeta(3)\right)C_A-\frac{14}{27}N_f\right] \\
&+ C_F^2\left[\frac{511}{64}-\frac{\pi^2}{16}-\frac{\pi^4}{60}-\frac{15}{4}\zeta(3)\right] \\
&+ C_FC_A\left[-\frac{1535}{192}-\frac{5\pi^2}{16}+\frac{7\pi^4}{720}+\frac{151}{36}\zeta(3)\right] \\
&+ C_FN_f\left[\frac{127}{96}+\frac{\pi^2}{24}+\frac{\zeta(3)}{18}\right]+\mathcal{O}\left(\frac{1}{N}\right)+\mathcal{O}\left(\frac{1}{M}\right). \qquad (4.4.2)
\end{aligned}$$

Up to order $\left(\frac{\alpha_s}{\pi}\right)^2$ the resummed coefficient function is provided by Eq. 4.3.10, at NLO the coefficient is:

$$\tilde{C}_{qq}^{\mathrm{ds},T,(1)}\left(N,M,\alpha_s(Q^2)\right) = 2A_1\mathcal{L}^2 - 2D_1\mathcal{L} + g_0^{(1)}\,, \qquad (4.4.3)$$

and by comparison with Eqs.(4.1.1,4.4.1) we obtain

$$A_{qq}^{(1)} = C_F\,, \qquad (4.4.4)$$

$$D_{\mathrm{ds},qq}^{(1)} = 0\,, \qquad (4.4.5)$$

$$g_{0,(\mathrm{ds},qq)}^{(1)} = C_F\left(\frac{\pi^2}{6}-4\right) = C_F\left(\zeta(2)-4\right)\,, \qquad (4.4.6)$$

where ds means double-soft. Hence, at order α_s^2 using $D_1 = 0$ we remain with

$$\tilde{C}_{qq}^{\mathrm{ds},T,(2)}\left(N,M,\alpha_s(Q^2)\right) = 2A_1^2\mathcal{L}^4 + \frac{4}{3}A_1\pi b_0\mathcal{L}^3 + 2A_1g_0^{(1)}\mathcal{L}^2 + 2A_2\mathcal{L}^2 - 2D_2\mathcal{L} + g_0^{(2)}\,, \qquad (4.4.7)$$

and by comparison with Eqs.(4.1.1,4.4.2) we obtain

$$A_{qq}^{(2)} = \frac{1}{2}C_F\left[\left(\frac{67}{18}-\frac{\pi^2}{6}\right)C_A-\frac{5}{9}N_f\right]\,, \qquad (4.4.8)$$

$$D_{\mathrm{ds},qq}^{(2)} = \frac{1}{2}C_F\left[\left(-\frac{101}{27}+\frac{7}{2}\zeta(3)\right)C_A+\frac{14}{27}N_f\right]\,, \qquad (4.4.9)$$

$$\begin{aligned}
g_{0,(\mathrm{ds},qq)}^{(2)} =\;& C_F^2\left[\frac{511}{64}-\frac{\pi^2}{16}-\frac{\pi^4}{60}-\frac{15}{4}\zeta(3)\right] \\
&+ C_FC_A\left[-\frac{1535}{192}-\frac{5\pi^2}{16}+\frac{7\pi^4}{720}+\frac{151}{36}\zeta(3)\right] \\
&+ C_FN_f\left[\frac{127}{96}+\frac{\pi^2}{24}+\frac{\zeta(3)}{18}\right]. \qquad (4.4.10)
\end{aligned}$$

Obviously, the resummation coefficients are the same of [29], except for a factor of 1/2 in the term D_2, as in their case the resummation formula includes a 1/2 multiplying the D terms. Furthermore, we note that the coefficients A are the coefficients of the cusp. In conclusion, we have proved the validity of the theoretical predictions given by Eq.4.2.1.

4.5 Single-soft limit

In this section, we present the novel contribution of this thesis, specifically, the single-soft coefficients up to NNLL for the SIDIS process in the qq-channel. Since the coefficient function in the single-soft limit contains significantly more terms than in the double-soft case, it is useful to introduce a convenient notation. We observe that, in the limit where one of the two Mellin variables approaches ∞, the coefficient function can be rewritten, e.g., in the x-single-soft limit, as follows:

$$\begin{aligned}
C_{qq}^{T}(\bar{N},\bar{M}) = 1 &+ \left(\frac{\alpha_s}{\pi}\right)\left(\ln^2\bar{N} f_2^{(1)}(M) + \ln\bar{N} f_1^{(1)}(M) + f_0^{(1)}(M)\right)\\
&+ \left(\frac{\alpha_s}{\pi}\right)^2\left(\ln^4\bar{N} f_4^{(2)}(M) + \ln^3\bar{N} f_3^{(2)}(M)\right.\\
&\left.+ \ln^2\bar{N} f_2^{(2)}(M) + \ln\bar{N} f_1^{(2)}(M) + f_0^{(2)}(M)\right)\\
&+ \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(\frac{1}{M}\right) + \mathcal{O}(\alpha_s^3)\,,
\end{aligned} \tag{4.5.1}$$

where $f_i^{(j)}(M)$ is a function only of the Mellin variable that does not approach ∞, where i denotes the power of $\ln N$ and j denotes the order of the perturbative expansion. The $\hat{z}$-single-soft case is entirely analogous; it suffices to exchange the roles of N and M.

The functions $f_i^{(j)}$ are listed in Appendix D.Furthermore, as before they were obtained using the packages [34] and [28] by applying the computational methods described in the previous sections,

Finally, we observe that the functions $f_0^{(i)}$ contain the constant terms. In particular, in the $\hat{x}$-single-soft limit, they include all terms of the form $\delta(1-\hat{x})\left[\frac{\ln^n(1-\hat{z})}{(1-\hat{z})}\right]_+$, $\delta(1-\hat{x})\delta(1-\hat{z})$, as well as the constants that arise when taking the asymptotic limit of the Harmonic sums with respect to N. Whereas, the other functions $f_j^{(i)}$ arise from the $\delta(1-\hat{z})$ contributions and the mixed contributions showed in Sec. 3.1. Obviously, the functions $f_j^{(i)}$ are different if we take either the $\hat{x}-$single-soft limit or the $\hat{z}-$single-soft limit.

4.5.1 $\hat{x}$-single-soft

Firstly, we consider the case with the scale $Q^2/(\bar{N}\bar{M})$. As we done in the previous section for the double soft case, taking the resummation formula expanded up to α_s^2 as in 4.3.11 and comparing it with the fixed order expression expressed as in 4.5.1, we obtain the relations:

$$A_1 = 2f_2^{(1)}(M)\,, \tag{4.5.2}$$

$$D_1(M) = -f_1^{(1)}(M) + A_1 \ln \bar{M}\,, \tag{4.5.3}$$

$$g_{0,qq}^{(1)}(M) = f_0^{(1)}(M) + D_1(M)\ln\bar{M} - \frac{A_1}{2}\ln^2\bar{M}\,, \tag{4.5.4}$$

$$f_4^{(2)}(M) - \frac{A_1^2}{8} = 0\,, \tag{4.5.5}$$

$$f_3^{(2)}(M) - \frac{A_1 b_0 \pi}{6} - \frac{D_1 A_1}{2} + \frac{A_1^2}{2}\ln\bar{M} = 0\,, \tag{4.5.6}$$

$$\begin{aligned} A_2(M) = 2f_2^{(2)}(M) - 2\Bigg(&-\frac{b_0\pi D_1}{2} + \frac{A_1 b_0 \pi}{2}\ln\bar{M} + \frac{D_1^2}{2} + \frac{A_1 g_0^{(1)}}{2} \\ &-\frac{3}{2}A_1 D_1 \ln\bar{M} + \frac{3}{4}A_1^2\ln\bar{M}\Bigg)\,, \end{aligned} \tag{4.5.7}$$

$$\begin{aligned} D_2(M) = -f_1^{(2)}(M) + \Bigg(&-b_0\pi D_1\ln\bar{M} + \frac{A_1 b_0\pi}{2}\ln^2\bar{M} - D_1 g_0^{(1)} + A_2\ln\bar{M} + D_1^2\ln\bar{M} \\ &+A_1 g_0^{(1)}\ln\bar{M} - \frac{3}{2}A_1 D_1 \ln^2\bar{M} + \frac{1}{2}A_1\ln^3\bar{M}\Bigg) \end{aligned} \tag{4.5.8}$$

$$\begin{aligned} g_0^{(2)}(M) = f_0^{(2)}(M) - \Bigg(&\frac{A_1^2\ln^4\bar{M}}{8} + \frac{A_1\pi\beta_0\ln^3\bar{M}}{6} - \frac{A_1 D_1\ln^3\bar{M}}{2} \\ &+\frac{A_1 g_0^{(1)}\ln^2\bar{M}}{2} + \frac{A_2\ln^2\bar{M}}{2} \\ &-\frac{\beta_0\pi D_1\ln^2\bar{M}}{2} + \frac{D_1^2\ln^2\bar{M}}{2} - D_1 g_0^{(1)}\ln\bar{M} - D_2\ln\bar{M}\Bigg)\,, \end{aligned} \tag{4.5.9}$$

for simplicity we omitted the qq subscript and the M dependence of the coefficients. Then, computing the above relations one obtains the single-soft coefficients

$$A_{qq}^{(1)} = C_F\,, \tag{4.5.10}$$

$$A_{qq}^{(2)} = \frac{1}{2}C_F\left[\left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f\right]\,, \tag{4.5.11}$$

$$D_{qq}^{(1)}(M) = \hat{P}_{qq}^{(0),T}(M)\,, \tag{4.5.12}$$

$$D_{qq}^{(2)}(M) = D_{\mathrm{ds},qq}^{(2)} - \pi\beta_0 F(M) + \hat{P}_{qq,\mathrm{NS}}^{(1),T}(M)\,, \tag{4.5.13}$$

$$g_{0,qq}^{(1)}(M) = g_{0,(\mathrm{ds},qq)}^{(1)} + F(M) = f_2^{(1)}(M)\ln^2\bar{M} - f_1^{(1)}(M)\ln\bar{M} + f_0^{(1)}(M)\,, \tag{4.5.14}$$

$$\begin{aligned} g_{0,qq}^{(2)}(M) = {}& f_4^{(2)}(M)\ln^4\bar{M} - f_3^{(2)}(M)\ln^3\bar{M} + f_2^{(2)}(M)\ln^2\bar{M} \\ &- f_1^{(2)}(M)\ln\bar{M} + f_0^{(2)}(M)\,. \end{aligned} \tag{4.5.15}$$

with

$$F(M) = C_F \ln \bar{M} \left(\ln \bar{M} + \frac{1}{2M^2 + 2M} - S_1(M) \right) + \frac{1}{2} C_F \left(S_1(M)^2 - \ln^2(\bar{M}) \right)$$
$$+ \frac{C_F \left(2M^2 - M - 1\right)}{2M^2(M+1)^2} + \frac{3}{2} C_F \left(S_2(M) - \zeta(2)\right) - \frac{C_F S_1(M)}{2M(M+1)} \,. \tag{4.5.16}$$

where S_i are the harmonic sums, while the time-like $\hat{P}^T$s are expressed in an explicit manner in the appendix C. Therefore, we have verified the theoretical prediction for the SIDIS process as reported in Eq. 4.2.9.

In particular, we have confirmed that by substituting the coefficients above into the expression 4.5.1 and taking the asymptotic limit for the variable $\bar{M}$, we recover the double-soft limit. In other words, we find the coefficient function in the double-soft limit as given by Eqs. 4.4.1 and 4.4.2. In fact, the coefficients in Eqs. 4.5.10-4.5.15 taking the asymptotic limit for M reduce to the double-soft coefficients.

Whereas, using the scale $Q^2/(\bar{N})$. Following the previous steps, but now taking the resummation formula expanded up to α_s^2 as in 4.3.12 we obtain the following resummation coefficients

$$A^{(1)}_{qq} = C_F \,, \tag{4.5.17}$$

$$A^{(2)}_{qq} = \frac{1}{2} C_F \left[\left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} N_f \right] , \tag{4.5.18}$$

$$D^{(1)}_{qq}(M) = \tilde{P}^{(0),T}_{qq}(M) \,, \tag{4.5.19}$$

$$D^{(2)}_{qq}(M) = D^{(2)}_{\mathrm{ds},qq} - \pi\beta_0 \tilde{F}(M) + \tilde{P}^{(1),T}_{qq,\mathrm{NS}}(M) \,, \tag{4.5.20}$$

$$g^{(1)}_{0,qq}(M) = g^{(1)}_{0,(\mathrm{ds},qq)} + \tilde{F}(M) = f^{(1)}_0(M) \,, \tag{4.5.21}$$

$$g^{(2)}_{0,qq}(M) = f^{(2)}_0(M) \tag{4.5.22}$$

where $\tilde{F}(M)$ is the non-logarithmic part of $F(M)$, while $\tilde{P}^{T,(i)}_{qq}$ is simply the Mellin transform of the time-like splitting function at order i without the constant terms.

As a final check, we have shown that the resummation formula with the scale $Q^2/\bar{N}$ and the one with $Q^2/\bar{N}\bar{M}$, using the respective coefficients reported above and their expansion up to α_s^2, namely Eq. 4.3.12 for $Q^2/\bar{N}$ and Eq. 4.3.11, produce the same result. Therefore, the two resummation formulas are equivalent.

4.5.2 $\hat{z}$-single-soft

In this section we provide the resummation coefficients for the $\hat{z}-$single-soft limit. Since the methods used to find the coefficients through the matching procedure with the fixed order are the same explained in the previous case, we only report the final results. Using the scale $Q^2/(\bar{N}\bar{M})$, we obtain

$$A_{qq}^{(1)} = C_F\,, \tag{4.5.23}$$

$$A_{qq}^{(2)} = \frac{1}{2}C_F\left[\left(\frac{67}{18}-\frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f\right], \tag{4.5.24}$$

$$D_{qq}^{(1)}(N) = \hat{P}_{qq,NS}^{(0)}(N) \tag{4.5.25}$$

$$D_{qq}^{(2)}(N) = D_{\mathrm{ds},qq}^{(2)} - \pi b_0 F(N) + \hat{P}_{qq,NS}^{(1)}(N) \tag{4.5.26}$$

$$g_{0,qq}^{(1)}(N) = g_{0,(\mathrm{ds},qq)}^{(1)} + F(N) = f_2^{(1)}(N)\ln^2\bar{N} - f_1^{(1)}(N)\ln\bar{N} + f_0^{(1)}(N)\,, \tag{4.5.27}$$

$$\begin{aligned} g_{0,qq}^{(2)}(M) = {}& f_4^{(2)}(N)\ln^4\bar{N} - f_3^{(2)}(N)\ln^3\bar{N} + f_2^{(2)}(N)\ln^2\bar{N} \\ & - f_1^{(2)}(N)\ln\bar{N} + f_0^{(2)}(N) \end{aligned} \tag{4.5.28}$$

with

$$\begin{aligned} F(N) = {}& C_F\ln\bar{N}\left(\ln\bar{N} + \frac{1}{2N^2+2N} - S_1(N)\right) + \frac{1}{2}C_F\left(S_1(N)^2 - \ln^2\bar{N}\right) \\ & - \frac{C_F S_1(N)}{2N^2+2N} + \frac{C_F(2N+1)}{2N^2(N+1)} + \frac{1}{2}C_F\left(\zeta(2) - S_2(N)\right) \end{aligned} \tag{4.5.29}$$

where S_i are the harmonic sums, while the space-like $\hat{P}$s are expressed in an explicit manner in the appendix C.

Whereas, using the scale $Q^2/\bar{N}$ we obtain

$$A_{qq}^{(1)} = C_F\,, \tag{4.5.30}$$

$$A_{qq}^{(2)} = \frac{1}{2}C_F\left[\left(\frac{67}{18}-\frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f\right], \tag{4.5.31}$$

$$D_{qq}^{(1)}(N) = \tilde{P}_{qq}^{(0)}(N)\,, \tag{4.5.32}$$

$$D_{qq}^{(2)}(N) = D_{\mathrm{ds},qq}^{(2)} - \pi\beta_0\tilde{F}(N) + \tilde{P}_{qq,\mathrm{NS}}^{(1)}(N)\,, \tag{4.5.33}$$

$$g_{0,qq}^{(1)}(N) = g_{0,(\mathrm{ds},qq)}^{(1)} + \tilde{F}(N) = f_0^{(1)}(N)\,, \tag{4.5.34}$$

$$g_{0,qq}^{(2)}(N) = f_0^{(2)}(N) \tag{4.5.35}$$

where $\tilde{F}(M)$ is the non-logarithmic part of $F(M)$, while $\tilde{P}_{qq}^{(i)}$ is simply the Mellin transform of the space-like splitting function at order i without the constant terms. Furthermore, as we done in the previous case, we verified that the resummation with the scale $Q^2/(\bar{N}\bar{M})$ is equal to the one with the scale $Q^2/\bar{M}$, and we have also checked the correct reduction to the double-soft limit case.

Conclusion

In this thesis, we have studied the SIDIS process, focusing specifically on the resummation of soft logarithms in the threshold limit. In particular, we have analyzed two distinct threshold regions: the double-soft limit, which corresponds to the elastic configuration where there is no momentum exchange between the incoming and outgoing partons, and the single-soft limit, which occurs when either the incoming or outgoing parton carries a fixed longitudinal momentum while the transferred momentum approaches its minimum value to allow this configuration. These two configurations correspond to the limits $\hat{x}, \hat{z} \to 1$ and either $\hat{x} \to 1$ with fixed $\hat{z}$ or $\hat{z} \to 1$ with fixed $\hat{x}$, respectively. In particular, the single-soft case is the novel result. The peculiarity of the asymmetric case is that, compared to the double-soft case, it also predicts all the next-to-leading power (NLP) corrections to the enhanced logarithms that emerge in the threshold limit.

Specifically, through a phase space argument we have verified that in both limits the coefficients function dependence is given in terms of a single soft scale, which is $Q^2(1-\hat{x})(1-\hat{z})$ in the double-soft case and either $Q^2(1-\hat{x})$ or $Q^2(1-\hat{z})$ in the single-soft cases. Which in Mellin space becomes $Q^2/\bar{N}\bar{M}$ and either $Q^2/\bar{N}$ or $Q^2/\bar{M}$. This result is completely analogous to the one obtained in the threshold limit for the DY at fixed rapidity [3], thereby showing the correspondence of the two process in the threshold limit. Thus we have obtained the resummation formula and its theoretical predictions for the SIDIS from the DY case [3]. Then through the fixed order results up to NNLO provided by [26] we have obtained the resummation coefficients up to NNLL in both soft limits for the NS case of the qq channel finding a perfect agreement with theoretical predictions.

A natural subsequent application is to extract from these results the first NLP corrections, namely soft logarithms suppressed by a factor N or M, in order to verify the predictions for this case established in [4]. Furthermore, as we have shown in Sec. 4.2, in the single-soft limit for the qq channel we have subtracted from the NS coefficient function terms which are represented by the diagrams of the type $q \to qq\bar{q}$, where there are two quark lines of the same flavour and one tags a q from one line and a q from the other. Therefore, a possible future development is to understand how to incorporate these terms into the resummation formula.

Appendix A

Analytic tools

A.1 Laplace transform

The Mellin transform is a specific case of the Laplace transform. Therefore, we begin by introducing the definition of the Laplace transform and some of its properties, which are also applicable in the Mellin case.

Def A.1. Given a function $f(t)$ with $t \in [0, \infty]$ its *monolateral Laplace trasnform* is defined as follows:

$$\tilde{f}(s) \equiv \mathcal{L}[f(t)](s) \equiv \int_0^\infty dte^{-ts} f(t) \tag{A.1}$$

The *inverse of the monolateral Laplace transform* is given by

$$f(t) \equiv \mathcal{L}^{-1}[\tilde{f}(s)](t) = \frac{1}{2\pi i} \int_{c+i\infty}^{c-i\infty} dse^{ts} \tilde{f}(s) \tag{A.2}$$

From Eq.(A.1) it follows immediately that if $f(t)$ is a real function, then $\tilde{f}(s)$ is real, hence we have the relation:

$$\tilde{f}(s^*) = \tilde{f}(s)^* \,, \tag{A.3}$$

where $*$ indicates complex conjugation.

A.2 Mellin Transform

The SIDIS coefficient functions dependence is given by the scaling variables $x, z \in [0, 1]$. When a function is defined in the range $[0, 1]$ the Laplace transform can be reduced to a Mellin transform through the change of variable $x = e^-t$

Def A.2. Given a function $f(x)$ with $x \in (0, 1)$ its *Mellin Transform* is defined as follows

$$\tilde{f}(N) \equiv \mathcal{M}[f(x)](N) \equiv \int_0^1 x^{N-1} f(x) \,. \tag{A.4}$$

Obviously, here we only renamed s as N, and then *the inverse of the Mellin transform* according to Eq.(A.2) is given by

$$f(t) \equiv \mathcal{M}^{-1}[\tilde{f}(N)](t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dNx^{-N} \tilde{f}(N) \tag{A.5}$$

Like in the Fourier and Laplace transforms exist a convolution product which factorize under Mellin transform, this *convolution* is defined as follows

$$
\begin{aligned}
(f \otimes g)(x) &= \int_x^1 dy f(y) g\left(\frac{x}{y}\right) \\
&= \int_0^1 dy \int_0^1 dz f(y) g(z) \delta(x - yz)\,, \qquad (A.6)
\end{aligned}
$$

and it can be extended to the many function case

$$
(f_1 \otimes \cdots \otimes f_n)(x) = \int_0^1 dy_1 \cdots \int_0^1 dy_n f_1(y_1) \ldots f_n(y_n) \delta(x - y_1 \ldots y_n)\,. \qquad (A.7)
$$

And from the second form of the convolution product we can note that $g \otimes f = f \otimes g$. Under Mellin transformation, as we expect, the convolution product factorize

$$
\begin{aligned}
\mathcal{M}[f \otimes g](N) &= \int_0^1 dx x^{N-1} \int_0^1 dy \int_0^1 dz f(y) g(z) \delta(x - yz) \\
&= \int_0^1 dy y^{N-1} f(y) \int_0^1 dz z^{N-1} g(z) = \tilde{f}(N) \tilde{g}(N)\,. \qquad (A.8)
\end{aligned}
$$

A.2.1 Plus distribution

From the cancellation of the collinear singularities in the coefficient function, plus distribution terms emerge. The plus distribution is defined as follows:

$$
\int_0^1 dz [f(z)]_+ g(z) \equiv \int_0^1 dz f(z) [g(z) - g(1)]\,. \qquad (A.9)
$$

Hence, from the definition, we obtain the following useful property

$$
\int_0^1 dz [f(z)]_+ = 0\,. \qquad (A.10)
$$

Furthermore, an equivalent definition for the plus distribution is provided in terms as the limit of a class of distributions:

$$
[f(z)]_+ = \lim_{\epsilon \to 0^+} \left[\Theta(1 - \epsilon - z) f(z) - \delta(1 - z) \int_0^{1-\epsilon} dy f(y) \right]\,. \qquad (A.11)
$$

where the limit is performed after the integration over the test function $g(z)$.

Plus distribution arise from $d-$dimensional regularized calculations from the identity

$$
x^{-1+\epsilon} = \frac{\delta(x)}{\epsilon} + \left(\frac{1}{x}\right)_+ + \epsilon^2 \left(\frac{\ln x}{x}\right)_+ + \mathcal{O}(\epsilon^2)\,, \qquad (A.12)
$$

for divergences in $x = 0$ or equivalently exchange x with $1 - x$ for $x = 1$. The above identity can be derived acting on a test function $g(x)$ as follows

$$
\begin{aligned}
\int_0^1 dx x^{-1+\epsilon} g(x) &= \int_0^1 dx x^{-1+\epsilon} [g(x) - g(0)] + \frac{g(0)}{\epsilon} \\
&= \int_0^1 dx \left[\frac{1}{x} + \epsilon \frac{\ln x}{x} + \mathcal{O}(\epsilon^2) \right] [g(x) - g(0)] + \frac{g(0)}{\epsilon} \\
&= \int_0^1 dx \left[\left(\frac{1}{x}\right)_+ + \epsilon \left(\frac{\ln x}{x}\right)_+ + \frac{\delta(x)}{\epsilon} \right] g(x) \qquad (A.13)
\end{aligned}
$$

$$
(1 - \hat{x})^{-1-\epsilon} = \frac{\delta(1 - \hat{x})}{\epsilon} + \sum_{i=0}^{\infty} \frac{1}{i!} \epsilon^i \left[\frac{\ln^i (1 - \hat{x})}{1 - \hat{x}} \right]_+\,, \qquad (A.14)
$$

A.2.2 Harmonic sums

In Sec. 3.1 we have shown that the Mellin transformation of a plus distribution can be expressed through polygamma functions $\psi^{(i)}$s. These functions can be viewed as the analytical continuation in the complex plan of the harmonic sums, this fact is showed in [28] and [27]. Here, we report the definition of the harmonic sums in order to introduce the notation for the results in the next appendices. A generic harmonic sums is defined as

$$S_{a_1,\ldots,a_k}(N) = \sum_{N\geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 1} \frac{\text{sign}(a_1)^{i_1}}{i_1^{|a_1|}} \cdots \frac{\text{sign}(a_k)^{i_k}}{i_k^{|a_k|}}\,, \tag{A.15}$$

where k is called the depth and $w = \sum_{i=0}^{k} |a_i|$ is called the weight of the harmonic sum $S_{a_1,\ldots,a_k}$. For instance, we report the explicit expression for the harmonic sums that appear more frequently in the calculation of the double- and single- soft limits, namely $k = 1, w = i > 0$

$$S_i(N) = \sum_{j=1}^{N} \frac{1}{j^i}\,, \tag{A.16}$$

then

$$S_1(N) = \sum_{j=1}^{N} \frac{1}{j} \qquad S_2(N) = \sum_{j=1}^{N} \frac{1}{j^2}\,. \tag{A.17}$$

In particular at large limit N we have

$$S_1(N) = \ln \bar{N} + \mathcal{O}\left(\frac{1}{N}\right) \qquad S_2(N) = \frac{\pi^2}{6} + \mathcal{O}\left(\frac{1}{N}\right)\,. \tag{A.18}$$

For completeness we provide the relation with $\psi^{(0)}$ and $\psi^{(1)}$ polygamma functions [27]

$$S_1(N) = \psi^{(0)}(N+1) + \gamma_E\,, \tag{A.19}$$

$$S_2(N) = -\psi^{(1)}(N+1) + \zeta(2)\,. \tag{A.20}$$

where $\zeta(i)$ represent the Riemann function evaluated at the point i, while γ_E is the Eulero-Mascheroni constant.

Appendix B

Running coupling and β values

We use the following expansion of the running strong coupling

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_R)}{l}\left[1 - \frac{\alpha_s(\mu_R)}{l} b_1 \ln(l) + \left(\frac{\alpha_s(\mu_R)}{l}\right)^2 \left(b_1^2 \left(\ln^2(l) - \ln(l) + l - 1\right) - b_2(l-1)\right)\right] + \mathcal{O}(\alpha_s(\mu_R)) \,, \tag{B.1}$$

where

$$l \equiv 1 + b_0 \alpha_s(\mu_R) \ln \frac{\mu^2}{\mu_R^2} \,, \tag{B.2}$$

and

$$\beta_0 = \frac{1}{12\pi}\left(11C_A - 2N_f\right) \,, \qquad \beta_1 = \frac{1}{24\pi^2}\left(17C_A^2 - 5C_A N_f - 3C_F N_f\right) \,,$$
$$\beta_2 = \frac{1}{64\pi^3}\left(\frac{2857}{54}C_A^3 - \frac{1415}{54}C_A^2 N_f - \frac{205}{18}C_A C_F N_f + C_F^2 N_f + \frac{79}{54}C_A N_f^2 + \frac{11}{9}C_F N_f^2\right) \,, \tag{B.3}$$

with N_f the number of flavors and

$$C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3} \,, \quad C_A = N_c = 3 \,. \tag{B.4}$$

Appendix C

NLO splitting functions

Here we reports the NS splitting function space-like and time-like up to NLO.

$$P_{qq} = \frac{\alpha_s}{\pi} P_{qq}^{(0)} + \left(\frac{\alpha_s}{\pi}\right)^2 P_{qq}^{(1)} + \mathcal{O}(\alpha_s^2) \tag{C.1}$$

Note that at leading order, the time-like case and space-like case are equivalent. Then, from [5]

$$P_{qq}^{(0)}(x) = P_{qq}^{T,(0)}(x) = \frac{C_F}{2}\left[\frac{1+x^2}{[1-x]_+} + \frac{3}{2}\delta(1-x)\right] , \tag{C.2}$$

that in Mellin space becomes

$$P_{qq}^{(0)}(N) = \frac{C_F}{2}\left[\frac{3}{2} + \frac{1}{N(N+1)} - 2S_1(N)\right] , \tag{C.3}$$

therefore

$$\hat{P}_{qq}^{(0)}(N) = C_F\left(\ln \bar{N} + \frac{1}{2N^2+2N} - S_1(N)\right) . \tag{C.4}$$

At NLO the difference between the NS singlet space-like splitting function and the NS time-like splitting function is given by [38]

$$\begin{aligned}\Delta_{qq,\mathrm{NS}}^{(1)}(x) = \frac{1}{16}C_F^2 \Bigg(\left(-32\left[\frac{1}{1-x}\right]_+ + 16x + 16\right) H_{1,0}(x) + \left(-32\left[\frac{1}{1-x}\right]_+ + 24x + 24\right) H_{0,0}(x) \\ + H_2(x)\left(-32\left[\frac{1}{1-x}\right]_+ + 16x + 16\right) + H_0(x)\left(24\left[\frac{1}{1-x}\right]_+ - 4x - 20\right)\Bigg) ,\end{aligned} \tag{C.5}$$

which in Mellin space becomes

$$\begin{aligned}\Delta_{qq,\mathrm{NS}}^{(1)}(N) = C_F^2 \Bigg(\frac{(2N+1)S_1(N)}{N^2(N+1)^2} + \frac{\left(3N^2+3N+2\right)S_2(N)}{2N(N+1)} - \frac{\left(3N^2+3N+2\right)\zeta(2)}{2N(N+1)} \\ - \frac{6N^3+9N^2+7N+2}{4N^3(N+1)^3} + 2\zeta(2)S_1(N) - 2S_1(N)S_2(N)\Bigg) .\end{aligned} \tag{C.6}$$

Whereas, the space-like splitting function in x space provided by [5] is

$$
\begin{aligned}
P^{(1)}_{qq,\mathrm{NS}}(x) = \frac{1}{4}\Bigg(& C_F^2 \left(\left(2H_0(x)H_1(x) - \frac{3H_0(x)}{2}\right)\left(2\left[\frac{1}{1-x}\right]_+ - x - 1\right) - \frac{1}{2}(x+1)H_0(x)^2 - \right.\\
& \left.\left(\frac{7x}{2}+\frac{3}{2}\right)H_0(x) - 5(1-x)\right)\\
& + C_F C_A\left(\left(\frac{1}{2}H_0(x)^2 + \frac{11H_0(x)}{6} - \frac{\pi^2}{6} + \frac{67}{18}\right)\left(2\left[\frac{1}{1-x}\right]_+ - x - 1\right)\right.\\
& \left. + (x+1)H_0(x) + \frac{20(1-x)}{3}\right)\\
& + \frac{1}{2}C_F N_F\left(\left(-\frac{2}{3}H_0(x) - \frac{10}{9}\right)\left(2\left[\frac{1}{1-x}\right]_+ - x - 1\right) - \frac{4(1-x)}{3}\right)\Bigg)\\
& + \frac{1}{4}\delta(1-x)\left(C_F^2\left(6\zeta(3)+\frac{3}{8}-\frac{\pi^2}{2}\right) - C_F C_A\left(-3\zeta(3)+\frac{17}{24}+\frac{11\pi^2}{18}\right)\right.\\
& \left. -\frac{1}{2}C_F N_F\left(\frac{1}{6}+\frac{2\pi^2}{9}\right)\right)\,,
\end{aligned}
\tag{C.7}
$$

then

$$
P^{(1),T}_{qq,\mathrm{NS}}(x) = P^{(1)}_{qq,\mathrm{NS}}(x) + \Delta^{(1)}_{qq,\mathrm{NS}}(x)\,. \tag{C.8}
$$

Whereas, the space-like splitting function in Mellin space becomes

$$
\begin{aligned}
P^{(1)}_{qq,\mathrm{NS}}(N) = C_F^2 \Bigg(& -\frac{(2N+1)S_1(N)}{2N^2(N+1)^2} - \frac{\left(3N^2+3N+2\right)S_2(N)}{4N(N+1)} + \frac{\left(3N^2+3N+2\right)\zeta(2)}{4N(N+1)}\\
& + \frac{3N^3+N^2-1}{4N^3(N+1)^3} - \zeta(2)S_1(N) + S_1(N)S_2(N) + S_3(N) - \zeta(3)\Bigg)\\
& + C_F N_F\left(-\frac{11N^2+5N-3}{36N^2(N+1)^2} + \frac{5S_1(N)}{18} - \frac{S_2(N)}{6} + \frac{\zeta(2)}{6}\right)\\
& + C_F C_A\Bigg(-\frac{\left(11N^2+11N+3\right)\zeta(2)}{12N(N+1)} + \frac{151N^4+236N^3+88N^2+3N+18}{72N^3(N+1)^3}\\
& + \frac{1}{2}\zeta(2)S_1(N) - \frac{67S_1(N)}{36} + \frac{11S_2(N)}{12} - \frac{S_3(N)}{2} + \frac{\zeta(3)}{2}\Bigg)\\
& + \frac{1}{4}\Bigg(C_F^2\left(6\zeta(3)+\frac{3}{8}-\frac{\pi^2}{2}\right)\\
& - C_F C_A\left(-3\zeta(3)+\frac{17}{24}+\frac{11\pi^2}{18}\right) - \frac{1}{2}\left(\frac{1}{6}+\frac{2\pi^2}{9}\right)C_F N_F\Bigg)\,,
\end{aligned}
\tag{C.9}
$$

then

$$
P^{(1),T}_{qq,\mathrm{NS}}(N) = P^{(1)}_{qq,\mathrm{NS}}(N) + \Delta^{(1)}_{qq,\mathrm{NS}}(N)\,. \tag{C.10}
$$

In conclusion we provide the explicit form of the $\hat{P}$ splitting function in both cases:

$$
\begin{aligned}
\hat{P}^{(1)}_{qq,NS}(N) = C_F^2 \Bigg(& -\frac{(2N+1)S_1(N)}{2N^2(N+1)^2} - \frac{\left(3N^2+3N+2\right)S_2(N)}{4N(N+1)} + \frac{\left(3N^2+3N+2\right)\zeta(2)}{4N(N+1)} \\
& + \frac{3N^3+N^2-1}{4N^3(N+1)^3} - \zeta(2)S_1(N) + S_1(N)S_2(N) + S_3(N) - \zeta(3) \Bigg) \\
& + C_F N_F \left(-\frac{5\ln\bar{N}}{18} - \frac{11N^2+5N-3}{36N^2(N+1)^2} + \frac{5S_1(N)}{18} - \frac{S_2(N)}{6} + \frac{\zeta(2)}{6} \right) \\
& + C_F C_A \Bigg(-\frac{1}{2}\zeta(2)\ln\bar{N} + \frac{67\ln\bar{N}}{36} - \frac{\left(11N^2+11N+3\right)\zeta(2)}{12N(N+1)} \\
& + \frac{151N^4+236N^3+88N^2+3N+18}{72N^3(N+1)^3} \\
& + \frac{1}{2}\zeta(2)S_1(N) - \frac{67S_1(N)}{36} + \frac{11S_2(N)}{12} - \frac{S_3(N)}{2} + \frac{\zeta(3)}{2} \Bigg) , \qquad \text{(C.11)}
\end{aligned}
$$

while, the time-like is

$$
\begin{aligned}
\hat{P}^{(1),T}_{qq}(N) = C_F^2 \Bigg(& \frac{(2N+1)S_1(N)}{2N^2(N+1)^2} + \frac{\left(3N^2+3N+2\right)S_2(N)}{4N(N+1)} - \frac{\left(3N^2+3N+2\right)\zeta(2)}{4N(N+1)} \\
& - \frac{3N^3+8N^2+7N+3}{4N^3(N+1)^3} + \zeta(2)S_1(N) - S_1(N)S_2(N) + S_3(N) - \zeta(3) \Bigg) \\
& + C_F N_F \left(-\frac{5\ln\bar{N}}{18} - \frac{11N^2+5N-3}{36N^2(N+1)^2} + \frac{5S_1(N)}{18} - \frac{S_2(N)}{6} + \frac{\zeta(2)}{6} \right) \\
& + C_F C_A \Bigg(-\frac{1}{2}\zeta(2)\ln\bar{N} + \frac{67\ln\bar{N}}{36} - \frac{\left(11N^2+11N+3\right)\zeta(2)}{12N(N+1)} \\
& + \frac{151N^4+236N^3+88N^2+3N+18}{72N^3(N+1)^3} \\
& + \frac{1}{2}\zeta(2)S_1(N) - \frac{67S_1(N)}{36} + \frac{11S_2(N)}{12} - \frac{S_3(N)}{2} + \frac{\zeta(3)}{2} \Bigg) . \qquad \text{(C.12)}
\end{aligned}
$$

Appendix D

Coefficient functions in soft limits

D.1 $\hat{x}$ single-soft limit

We report the coefficient function C^T_{qq} in Mellin space and in the $\hat{x}$ single-soft limit using the notation of Eq.(4.5.1).

NLO results

$$f_2^{(1)}(M) = \frac{C_F}{2} \tag{D.1}$$

$$f_1^{(1)}(M) = C_F\left(S_1(M) - \frac{2}{4M^2+4M}\right) \tag{D.2}$$

$$\begin{aligned} f_0^{(1)}(M) = &\frac{C_F\left(2M^2-M-1\right)}{2M^2(M+1)^2} + \frac{1}{2}C_F S_1(M)^2 - \frac{C_F S_1(M)}{2M(M+1)} \\ &+ \frac{3}{2}C_F S_2(M) - \frac{C_F\zeta(2)}{2} - 4C_F \end{aligned} \tag{D.3}$$

NNLO results

$$f_4^{(2)}(M) = \frac{1}{8}C_F^2\,, \tag{D.4}$$

$$f_3^{(2)}(M) = C_F^2\left(\frac{S_1(M)}{2} - \frac{1}{4M(M+1)}\right) + \frac{11C_F C_A}{72} - \frac{C_F N_f}{36}\,, \tag{D.5}$$

$$\begin{aligned} f_2^{(2)}(M) = &C_F^2\left(\frac{4M^2-2M-1}{8M^2(M+1)^2} + \frac{3}{4}S_1(M)^2 - \frac{3S_1(M)}{4M(M+1)} + \frac{3S_2(M)}{4} - \frac{\zeta(2)}{4} - 2\right) \\ &+ C_F C_A\left(\frac{11S_1(M)}{24} - \frac{11}{48M(M+1)} - \frac{\zeta(2)}{4} + \frac{67}{72}\right) \\ &+ C_F N_f\left(-\frac{1}{12}S_1(M) + \frac{1}{24M(M+1)} - \frac{5}{36}\right)\,, \end{aligned} \tag{D.6}$$

$$
\begin{aligned}
f_1^{(2)}(M) = C_{\mathrm{F}}^2 \Bigg(& -\frac{\left(3M^2+3M+5\right) S_2(M)}{4M(M+1)} + \frac{3\left(M^2+M+1\right)\zeta(2)}{4M(M+1)} \\
& -\frac{\left(16M^4+32M^3+12M^2+6M+3\right) S_1(M)}{4M^2(M+1)^2} + \frac{8M^4+19M^3+14M^2+8M+4}{4M^3(M+1)^3} \\
& -\frac{3}{2}\zeta(2)S_1(M) + \frac{1}{2}S_1(M)^3 - \frac{3S_1(M)^2}{4M(M+1)} + \frac{5}{2}S_2(M)S_1(M) - S_3(M) + \zeta(3)\Bigg) \\
& + C_{\mathrm{F}} N_f \Bigg(-\frac{\left(10M^2+10M-3\right) S_1(M)}{36M(M+1)} - \frac{1}{12}S_1(M)^2 - \frac{S_2(M)}{12} \\
& + \frac{5M+8}{36M(M+1)^2} + \frac{\zeta(2)}{12} - \frac{7}{27}\Bigg) \\
& + C_{\mathrm{F}} C_{\mathrm{A}} \Bigg(\frac{\left(134M^2+134M-33\right) S_1(M)}{72M(M+1)} - \frac{85M^4+203M^3+154M^2+36M+18}{72M^3(M+1)^3} \\
& -\frac{1}{2}\zeta(2)S_1(M) + \frac{11}{24}S_1(M)^2 + \frac{11S_2(M)}{24} \\
& + \frac{S_3(M)}{2} + \frac{\zeta(2)}{4M(M+1)} - \frac{11\zeta(2)}{24} - \frac{9\zeta(3)}{4} + \frac{101}{54}\Bigg) \,,
\end{aligned}
\tag{D.7}
$$

$$\tag{D.8}$$

$$
\begin{aligned}
f_0^{(2)}(M) = C_{\mathrm{F}}^2 \Bigg(& \frac{1}{8}S_1(M)^4 - \frac{S_1(M)^3}{4M(M+1)} - \frac{5}{4}\zeta(2)S_1(M)^2 \\
& + \frac{\left(-M^5+13M^4+41M^3+15M^2+2\right) S_1(M)}{8M^3(M+1)^3} + \frac{5\zeta(2)S_1(M)}{4M^2+4M} - \frac{1}{2}\zeta(3)S_1(M) \\
& + \frac{S_1(M)^2\left(7M^2(M+1)^2S_2(M) - 2\left(4M^4+8M^3+3M^2+2M+1\right)\right)}{4M^2(M+1)^2} \\
& + \frac{\left(-7M(M+1)S_2(M)\right) S_1(M)}{4M^2(M+1)^2} + \frac{1}{2}\left(5S_3(M) - 2S_{2,1}(M)\right) S_1(M) - \frac{31\zeta(2)^2}{40} \\
& + \frac{-33M^6-50M^5+29M^4+81M^3+65M^2+48M+15}{8M^4(M+1)^4} \\
& + \frac{\left(4M^2-2M-1\right)\zeta(2)}{8M^2(M+1)^2} + \frac{\left(19M^4+38M^3+12M^2+M+1\right)\zeta(2)}{4M^2(M+1)^2} + \frac{13\zeta(2)}{8} \\
& - \frac{\zeta(3)}{2M(M+1)} - \frac{15\zeta(3)}{4} \\
& + \frac{3\left(7\zeta(3)M^2+7\zeta(3)M+2\zeta(3)\right)}{8M(M+1)} - \frac{\left(27M^4+54M^3+18M^2+7M+5\right) S_2(M)}{4M^2(M+1)^2} \\
& - \frac{5}{4}\zeta(2)S_2(M) - \frac{\left(9M^2+9M+10\right) S_3(M)}{8M(M+1)} - \frac{33S_4(M)}{8} - \frac{\left(3M^2+3M-2\right) S_{2,1}(M)}{4M(M+1)}
\end{aligned}
$$

$$+\frac{13}{4}S_{2,2}(M) - 2S_{3,1}(M) + \frac{3}{2}S_{2,1,1}(M) + \frac{511}{64}\Big)$$

$$+\frac{1}{36}(11C_{\rm A} - 2N_f)\zeta(3)C_{\rm F}$$

$$+ N_f C_{\rm F}\Bigg(-\frac{1}{36}S_1(M)^3 - \frac{\left(10M^2+10M-3\right)S_1(M)^2}{72M(M+1)}$$

$$-\frac{\left(28M^3+56M^2+13M-24\right)S_1(M)}{108M(M+1)^2} + \frac{1}{12}\zeta(2)S_1(M) - \frac{1}{12}S_2(M)S_1(M)$$

$$-\frac{11M^4-74M^3-109M^2-6M+9}{216M^3(M+1)^3} + \frac{\left(5M^2+5M-3\right)\zeta(2)}{36M(M+1)}$$

$$+\frac{\zeta(2)}{24M(M+1)} + \frac{7\zeta(2)}{18} + \frac{\zeta(3)}{12} - \frac{\left(20M^2+20M-3\right)S_2(M)}{72M(M+1)} + \frac{S_3(M)}{36} + \frac{127}{96}\Bigg)$$

$$+ C_{\rm A}C_{\rm F}\Bigg(\frac{11}{72}S_1(M)^3 + \frac{\left(404M^5+1239M^4+1038M^3-70M^2-273M+54\right)S_1(M)}{216M^2(M+1)^3}$$

$$-\frac{11}{24}\zeta(2)S_1(M) - \frac{13}{4}\zeta(3)S_1(M)$$

$$-\frac{S_1(M)^2\left(-134M^2+36(M+1)S_2(M)M-134M+33\right)}{144M(M+1)}$$

$$\frac{-6S_1(M)\left(11M^2+11M+6\right)S_2(M)}{144M(M+1)} - \frac{1}{2}\left(S_3(M) - 2S_{2,1}(M)\right)S_1(M) + \frac{21\zeta(2)^2}{20} +$$

$$\frac{124M^6-1044M^5-2679M^4-2396M^3-1002M^2-765M-270}{432M^4(M+1)^4}$$

$$-\frac{11\zeta(2)}{48M(M+1)} - \frac{101\zeta(2)}{36} - \frac{29\zeta(2)M^4+58\zeta(2)M^3+17\zeta(2)M^2-30\zeta(2)M-9\zeta(2)}{36M^2(M+1)^2}$$

$$+\frac{\left(11M^2+11M+39\right)\zeta(3)}{24M(M+1)} + \frac{43\zeta(3)}{12} + \frac{\left(250M^3+250M^2-69M-36\right)S_2(M)}{144M^2(M+1)}$$

$$-\frac{1}{2}\zeta(2)S_2(M) - \frac{\left(11M^2+11M-18\right)S_3(M)}{72M(M+1)} + S_4(M) - \frac{S_{2,1}(M)}{2M(M+1)}$$

$$+S_{2,2}(M) + S_{3,1}(M) - \frac{3}{2}S_{2,1,1}(M) - \frac{1535}{192}\Bigg) \tag{D.9}$$

D.2 $\hat{z}$ single-soft limit

We report the coefficient function C_{qq}^T in Mellin space and in the $\hat{z}$ single-soft limit using the notation of Eq.(4.5.1).

NLO results

$$f_2^{(1)}(N) = \frac{C_F}{2} \tag{D.10}$$

$$f_1^{(1)}(N) = C_F\left(S_1(N) - \frac{1}{2N^2+2N}\right) \tag{D.11}$$

$$f_0^{(1)}(N) = C_F\left(\frac{2N+1}{2N^2(N+1)} + \frac{1}{2}S_1(N)^2 - \frac{S_1(N)}{2N(N+1)} - \frac{S_2(N)}{2} + \frac{3\zeta(2)}{2} - 4\right) \tag{D.12}$$

NNLO results:

$$f_4^{(2)}(N) = \frac{1}{8}C_F^2 \tag{D.13}$$

$$f_3^{(2)}(N) = C_F^2\left(\frac{S_1(N)}{2} - \frac{1}{4N(N+1)}\right) + \frac{11}{72}C_FC_A - \frac{1}{36}C_FN_F \tag{D.14}$$

$$\begin{aligned} f_2^{(2)}(N) = {} & C_F^2\left(\frac{4N^2+6N+3}{8N^2(N+1)^2} + \frac{3}{4}S_1(N)^2 - \frac{3S_1(N)}{4N(N+1)} - \frac{S_2(N)}{4} + \frac{3\zeta(2)}{4} - 2\right) \\ & + C_FC_A\left(\frac{11S_1(N)}{24} - \frac{11}{48N(N+1)} - \frac{\zeta(2)}{4} + \frac{67}{72}\right) \\ & + C_FN_F\left(-\frac{1}{12}S_1(N) + \frac{1}{24N(N+1)} - \frac{5}{36}\right) \end{aligned} \tag{D.15}$$

$$\begin{aligned} f_1^{(2)}(N) = {} & C_F^2\Bigg(\frac{3\left(N^2+N+1\right)S_2(N)}{4N(N+1)} - \frac{\left(3N^2+3N+5\right)\zeta(2)}{4N(N+1)} + \frac{8N^3+13N^2+5N-3}{4N^2(N+1)^3} \\ & - \frac{\left(16N^4+32N^3+12N^2-10N-5\right)S_1(N)}{4N^2(N+1)^2} + \frac{5}{2}\zeta(2)S_1(N) + \frac{1}{2}S_1(N)^3 \\ & - \frac{3S_1(N)^2}{4N(N+1)} - \frac{3}{2}S_2(N)S_1(N) - S_3(N) + \zeta(3)\Bigg) \\ & + C_FN_F\Bigg(-\frac{\left(10N^2+10N-3\right)S_1(N)}{36N(N+1)} + \frac{5N^2-4N-6}{36N^2(N+1)^2} \\ & - \frac{1}{12}S_1(N)^2 + \frac{S_2(N)}{4} - \frac{\zeta(2)}{4} - \frac{7}{27}\Bigg) \\ & + C_FC_A\Bigg(\frac{\left(134N^2+134N-33\right)S_1(N)}{72N(N+1)} - \frac{85N^4+71N^3-44N^2-30N+18}{72N^3(N+1)^3} \\ & - \frac{1}{2}\zeta(2)S_1(N) + \frac{11}{24}S_1(N)^2 - \frac{11S_2(N)}{8} + \frac{S_3(N)}{2} + \frac{11N\zeta(2)}{6(N+1)} + \frac{\zeta(2)}{4N(N+1)} \\ & + \frac{11\zeta(2)}{6(N+1)} - \frac{11\zeta(2)}{24} - \frac{9\zeta(3)}{4} + \frac{101}{54}\Bigg) \end{aligned} \tag{D.16}$$

$$
\begin{aligned}
f_0^{(2)}(N) = C_F^2 \Bigg(& \frac{1}{8} S_1(N)^4 - \frac{S_1(N)^3}{4N(N+1)} + \frac{7}{4}\zeta(2) S_1(N)^2 + \frac{\left(N^4+17N^3+4N^2-24N-8\right) S_1(N)}{8N^3(N+1)^2} \\
& - \frac{3\zeta(2)S_1(N)}{4N(N+1)} - \frac{\zeta(2)S_1(N)}{N^2+N} - \frac{7}{2}\zeta(3)S_1(N) \\
& - \frac{\left(8N^4+16N^3+5(N+1)^2 S_2(N) N^2 + 6N^2 - 6N - 3\right) S_1(N)^2}{4N^2(N+1)^2} \\
& + \frac{5S_2(N)S_1(N)}{4N(N+1)} + \frac{3}{2} S_3(N) S_1(N) + S_{2,1}(N) S_1(N) \\
& - \frac{11\zeta(2)^2}{40} + \frac{2N^7 - 25N^6 - 106N^5 - 142N^4 - 83N^3 - 21N^2 - 13N - 5}{8N^4(N+1)^4} \\
& + \frac{\left(4N^2+6N+3\right)\zeta(2)}{8N^2(N+1)^2} - \frac{\left(20N^3+40N^2+13N-3\right)\zeta(2)}{4N(N+1)^2} + \frac{13\zeta(2)}{8} - \frac{\zeta(3)}{2N(N+1)} - \frac{15\zeta(3)}{4} \\
& - \frac{3\left(\zeta(3)N^2+\zeta(3)N-6\zeta(3)\right)}{8N(N+1)} + \frac{\left(12N^3+24N^2+7N-1\right) S_2(N)}{4N(N+1)^2} + \frac{3}{4}\zeta(2) S_2(N) \\
& - \frac{3\left(3N^2+3N+2\right) S_3(N)}{8N(N+1)} + \frac{23 S_4(N)}{8} + \frac{\left(3N^2+3N-2\right) S_{2,1}(N)}{4N(N+1)} \\
& - \frac{1}{4} S_{2,2}(N) - \frac{1}{2} S_{3,1}(N) - \frac{3}{2} S_{2,1,1}(N) + \frac{511}{64} \Bigg) \\
& + \frac{1}{36}(11C_A - 2N_F)\zeta(3) C_F \\
& + C_F \Bigg(\frac{S_1(N)^2 \left(N^2+N+2\right)^2}{16N^2(N+1)^2\left(N^2+N-2\right)} + \frac{\zeta(2)\left(N^2+N+2\right)^2}{16N^2(N+1)^2\left(N^2+N-2\right)} \\
& - \frac{S_2(N)\left(N^2+N+2\right)^2}{16N^2(N+1)^2\left(N^2+N-2\right)} \\
& + \frac{N^{10}+8N^9+33N^8+127N^7+459N^6+1111N^5}{16(N-1)N^4(N+1)^4(N+2)^3} \\
& + \frac{1725N^4+1776N^3+1192N^2+464N+80}{16(N-1)N^4(N+1)^4(N+2)^3} \\
& - \frac{\left(N^6+12N^5+53N^4+86N^3+80N^2+56N+16\right) S_1(N)}{8(N-1)N^3(N+1)^3(N+2)^2} \Bigg) \\
& + N_F C_F \Bigg(-\frac{1}{36} S_1(N)^3 - \frac{\left(10N^2+10N-3\right) S_1(N)^2}{72N(N+1)} - \frac{1}{12}\zeta(2) S_1(N) + \frac{1}{12} S_2(N) S_1(N) \\
& - \frac{7N^2 S_1(N)}{27(N+1)^2} - \frac{14N S_1(N)}{27(N+1)^2} - \frac{S_1(N)}{9N(N+1)^2} - \frac{S_1(N)}{6N^2(N+1)^2} - \frac{13 S_1(N)}{108(N+1)^2}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{11N^4+46N^3-19N^2-90N-45}{216N^3(N+1)^3}+\frac{\zeta(2)}{24N(N+1)}-\frac{\zeta(2)}{36}+\frac{\zeta(3)}{12}\\
&+\frac{\left(20N^2+20N-3\right)S_2(N)}{72N(N+1)}-\frac{11S_3(N)}{36}+\frac{1}{6}S_{2,1}(N)+\frac{127}{96}\Bigg)\\
&+C_FC_A\left(\frac{11NS_1(N)^3}{72(N+1)}+\frac{11S_1(N)^3}{72(N+1)}-\frac{3}{4}\zeta(2)S_1(N)^2\right.\\
&+\frac{\left(404N^5+781N^4+122N^3-12N^2+144N-54\right)S_1(N)}{216N^3(N+1)^2}\\
&+\frac{11N\zeta(2)S_1(N)}{24(N+1)}+\frac{3\zeta(2)S_1(N)}{4N(N+1)}+\frac{11\zeta(2)S_1(N)}{24(N+1)}+\frac{5}{4}\zeta(3)S_1(N)\\
&+\frac{\left(S_1(N)\left(134N^2+72(N+1)S_2(N)N+134N-33\right)\right)S_1(N)}{144N(N+1)}\\
&-\frac{\left(6\left(11N^2+11N+12\right)S_2(N)\right)S_1(N)}{144N(N+1)}\\
&-\frac{108N^7+308N^6-348N^5-999N^4-298N^3+324N^2-153N-162}{432N^4(N+1)^4}\\
&-\frac{3}{2}S_{2,1}(N)S_1(N)-\frac{19\zeta(2)^2}{20}+\frac{\left(70N^4+140N^3+49N^2-15N-6\right)\zeta(2)}{24N^2(N+1)^2}\\
&-\frac{11\zeta(2)}{48N(N+1)}-\frac{101\zeta(2)}{36}+\frac{\left(11N^2+11N-15\right)\zeta(3)}{24N(N+1)}+\frac{43\zeta(3)}{12}\\
&-\frac{\left(286N^3+572N^2+181N-33\right)S_2(N)}{144N(N+1)^2}-\frac{1}{4}\zeta(2)S_2(N)+\frac{121S_3(N)}{72}-\frac{5S_4(N)}{4}\\
&\left.-\frac{\left(11N^2+11N-9\right)S_{2,1}(N)}{12N(N+1)}-\frac{1}{2}S_{2,2}(N)+2S_{2,1,1}(N)-\frac{1535}{192}\right)
\end{aligned}
\tag{D.17}
$$

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